Lifting Automorphisms of $K_0(M_{n-1}(\mathbb{C}) \otimes \mathcal{O}_n)$

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Abstract

K-theory induces a homomorphism from the automorphism group $\mathrm{Aut}(A)$ of an algebra A to the automorphism group $\mathrm{Aut}(K_0(A))$ by sending an algebra automorphism φ of A to its induced automorphism φ_* of $K_0(A)$. This homomorphism is known to be surjective for special kinds of C*-algebras (like $M_{n-1}(\mathbb{C}) \otimes \mathcal{O}_n$, where $\mathcal{O}_n$ is the n-th Cuntz algebra), but not by explicit construction. In this paper, we provide an explicit proof of this result when $A = M_{n-1}(\mathbb{C}) \otimes \mathcal{O}_n$ by constructing an automorphism φ of A for each automorphism ψ of $K_0(A)$ so that $\varphi_* = \psi$. As the methods used are purely algebraic, this result also holds when $A = M_{n-1}(F) \otimes L_n(F)$, where $L_n(F)$ is the Leavitt algebra of module type $(1, n-1)$ over a field F. Lastly, we show that these methods can produce algebra automorphisms φ with the same order as their induced automorphisms φ_* under certain circumstances.

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Introduction

The following thesis is written so that an undergraduate familiar with elementary abstract and linear algebra should be able to understand the entirety of the paper with some minor omissions.

Part 1 of the paper will discuss definitions and theorems necessary for understanding the problem statement and main results presented in the second part. We begin by defining an algebra over a field, move on to defining the Leavitt and Cuntz algebras, and finish with a discussion of lower K-theory of C*-algebras in terms of projections. This last section could be rewritten for Leavitt algebras by replacing projections with idempotents and modifying the proofs accordingly. We will occasionally give nothing but a citation as proof for some results, as a full proof would be a significant detour from the intended subject of the thesis.

Part 2 begins with a statement of the main problem from which a simplified, but equivalent version of the problem is derived. After a brief justification is given for why the problem is of interest, the two main results and their proofs are presented. Most of these proofs rely solely on calculation and will be better understood if the reader performs these calculations along with the author. The section finishes by showing how the two major theorems of the paper relate to one another.

1 Prerequisite Concepts

1.1 Universal algebras

Firstly, before defining the important Leavitt and Cuntz algebras, we define what an algebra is.

Definition 1.1. Let V be a vector space over a field F. An *algebra* $A = \langle V, \cdot \rangle$ over a field F is a vector space V equipped with a bilinear product $(a, b) \mapsto a \cdot b$.

That is, for all $\lambda_1, \dots, \lambda_4 \in F$ and $v_1, \dots, v_4 \in V$, the following identity holds:

$$
\begin{aligned}
(\lambda_1 v_1 + \lambda_2 v_2) \cdot (\lambda_3 v_3 + \lambda_4 v_4) &= \lambda_1 v_1 \cdot (\lambda_3 v_3 + \lambda_4 v_4) + \lambda_2 v_2 \cdot (\lambda_3 v_3 + \lambda_4 v_4) \\
&= \lambda_1 \lambda_3 (v_1 \cdot v_3) + \lambda_1 \lambda_4 (v_1 \cdot v_4) + \lambda_2 \lambda_3 (v_2 \cdot v_3) + \lambda_2 \lambda_4 (v_2 \cdot v_4).
\end{aligned}
$$

Note that the bilinear product on A need not be associative or commutative! However, one usually assumes associativity unless specified otherwise. For clarity, though, we shall state that A is an *associative* or a *commutative* algebra if these descriptions are accurate. Also noteworthy is that an algebra need not have a multiplicative identity; algebras with a multiplicative identity are called *unital*.

We provide two examples with which the reader is certainly already familiar.

Example 1.2. Let $C = \mathbb{R}^2$ be an algebra over $\mathbb{R}$ where scalar multiplication is given by

$$\lambda(a, b) = (\lambda a, \lambda b)$$

for all $\lambda \in \mathbb{R}$, vector addition is given by

$$(a, b) + (c, d) = (a + c, b + d)$$

and vector multiplication is given by

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc)$$

for all $(a, b), (c, d) \in C$. It should at this point be clear that C is isomorphic to the algebra of complex numbers $\mathbb{C}$ over the field $\mathbb{R}$. Similar constructions can be made for the quaternions $\mathbb{H}$ and octonions $\mathbb{O}$.

Example 1.3. Let $n \in \mathbb{Z}_{>0}$ and let $M_n(\mathbb{C})$ be the set of $n \times n$ matrices with complex entries. Since matrix multiplication has an identity, is associative, and distributes with respect to addition, $M_n(\mathbb{C})$ is a unital associative algebra over $\mathbb{C}$ when equipped with the standard operations associated with matrices.

We shall later define both the Leavitt and Cuntz algebras as generated by certain elements and constrained by specific relations. Hence, we find it useful to discuss what the presentation of an associative algebra is. In order to do so, we first must define the free left module on a set X, the free algebra on a set X, and finally presentations of algebras. We preface these definitions with the fact that a *module* is analogous to a vector space taken over a ring instead of a field.

Definition 1.4. Let X be a set and let R be a ring. The *free left module* on X over R is the module $M = \bigoplus_{x \in X} R$. That is, a family $m = (m_x)_{x \in X}$ in R indexed by X is an element of M if and only if there exists a finite subset $S \subset X$ such that $m_x \neq 0$ if and only $x \in S$.

Example 1.5. Let $X = \{a, b, c\}$ and let $R = \mathbb{Z}$. Then the free left module M on X over R is the set of all formal linear combinations

$$n_1 a + n_2 b + n_3 c$$

where $n_1, n_2, n_3 \in \mathbb{Z}$. Scalar multiplication is given by

$$k(n_1 a + n_2 b + n_3 c) = kn_1 a + kn_2 b + kn_3 c$$

and vector addition is given by

$$(n_1a + n_2b + n_3c) + (m_1a + m_2b + m_3c) = (n_1 + m_1)a + (n_2 + m_2)b + (n_3 + m_3)c,$$

where everything but a, b, and c is an element of $\mathbb{Z}$.

When X is infinite, one should think of M as one might think of an infinite dimensional vector space, remembering that M is over a ring instead of a field.

Recall that a monoid is like a group, but does not respect the inverse axiom. This is needed for our definition of a free algebra.

Definition 1.6. Let X be a set and let F be a field. Define a free monoid

$$X^\infty = \bigcup_{i=1}^{\infty} \left(\prod_{j=1}^{i} X \right),$$

where multiplication is given by concatenation; that is, the product of $x_1, x_2 \in X^\infty$ is simply x_1x_2. Define the *free algebra* A on X over a field F to be the free left module on X^∞ over F, where multiplication on A is inherited from X^∞. That is, given $x_1, \cdots, x_n, y_1, \ldots, y_m \in X^\infty$ and coefficients $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m \in F$, we have

$$\left(\sum_{i=1}^{n} \lambda_i x_i \right) \left(\sum_{j=1}^{m} \mu_j y_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} (\lambda_i \mu_j) x_i y_j.$$

The multiplication on X^∞ is associative and has an identity, so every free algebra is a unital associative algebra. Now that we have defined the free algebra on a generating set X over a field F, we can define what a presentation of an algebra is.

Definition 1.7. Let A be the free algebra on X over a field F and let R be a subset of A. The *universal associative algebra* B generated by X and subject to the relations $r = 0$ for all $r \in R$ is the quotient algebra $B = A/\langle R \rangle$, where $\langle R \rangle$ is the ideal generated by the elements of R.

There is nothing fundamentally different about the presentation of an algebra and the presentation of a group. Before delving into some examples, we prove a theorem which will help us connect these presentations to algebras we already know.

Theorem 1.8. Let A be the free F-algebra generated by a set X, let B be the universal associative F-algebra generated by X and subject to relations $R = \{r_\alpha \colon \alpha \in I\}$, and let C be a unital associative F-algebra. Given a map $f \colon X \to C$, one can define an algebra homomorphism $\varphi \colon A \to C$ by setting

$$\varphi\big(\lambda(x_1 x_2 \dots x_n)\big) = \lambda \prod_{i=1}^{n} f(x_i)$$

for any string $x_1 x_2 \dots x_n \in X^\infty$ and $\lambda \in F$, and then extending linearly.

If $\varphi(r_\alpha) = 0$ for all $\alpha \in I$, then there exists an algebra homomorphism $\overline{\varphi} : B \to C$ such that $\varphi = \overline{\varphi} \circ \pi$, where $\pi \colon A \to B$ is the quotient map. Furthermore, if $\text{Ker}(\varphi) = \langle R \rangle$, then $\overline{\varphi}$ is injective and if $f(X)$ generates B, then φ is surjective.

This result turns out to be fairly self-evident, as the proof shall show.

Proof. Elements of B are cosets of the form $a + \langle R \rangle$, where $a \in A$, so define $\overline{\varphi} \colon B \to C$ by $\overline{\varphi}(a + \langle R \rangle) = \varphi(a)$. We first claim that φ is well-defined. Let $a, a' \in A$ and suppose that $a + \langle R \rangle = a' + \langle R \rangle$. Thus $a' \in a + \langle R \rangle$, so there exists $x \in \langle R \rangle$ such that $a' = a + x$. Then

$$\overline{\varphi}(a' + \langle R \rangle) = \varphi(a') = \varphi(a + x) = \varphi(a) + 0 = \overline{\varphi}(a + \langle R \rangle),$$

where $\varphi(a + x) = \varphi(a) + 0$ because we assumed that φ vanishes on the generators of $\langle R \rangle$. Hence $\overline{\varphi}$ is well-defined. That $\overline{\varphi}$ is also an algebra homomorphism follows directly from the fact that φ and π are algebra homomorphisms.

Now suppose that $\text{Ker}(\varphi) = \langle R \rangle$; we claim $\overline{\varphi}$ is injective. Let $b \in B$ and suppose that $\overline{\varphi}(b) = 0$. Choose $a \in A$ such that $b \in a + \langle R \rangle$. Then $\overline{\varphi}(b) = \varphi(a) = 0$, so $a \in \langle R \rangle$. Since $\langle R \rangle$ is closed under addition, $a + \langle R \rangle = \langle R \rangle$, meaning $b = 0$. Hence $\overline{\varphi}$ is injective.

Now suppose that $f(X)$ generates C; we claim φ is surjective. Let $c \in C$. Since $f(X)$ generates C, there exists string lengths $l_1, \dots, l_n \in \mathbb{Z}_{\geq 0}$, elements $x_{i,1}, \dots, x_{i,l_i} \in X$ for all $i \in \{1, \dots, n\}$, and coefficients $\lambda_1, \dots, \lambda_n \in F$ such that

$$c = \sum_{i=1}^{n} \lambda_i \prod_{j=1}^{l_i} f(x_{i,j}).$$

Set

$$a = \sum_{i=1}^{n} \lambda_i \prod_{j=1}^{l_i} x_{i,j}.$$

Then $\overline{\varphi}(a + \langle R \rangle) = \varphi(a) = c$. Hence $\overline{\varphi}$ is surjective. $\square$

We now give a definition which doubles as an example.

Example 1.9. Let F be a field and let $M_n(F)$ be the universal unital associative algebra over F generated by the set

$$X = \{f_{1,1}, f_{1,2}, \ldots, f_{1,n}, f_{2,1}, \ldots, f_{n,1}\}$$

and subject to the relations

$$f_{1,i} \cdot f_{1,j} = 0 \qquad \text{if } i > 1 \tag{1.1}$$

$$f_{i,1} \cdot f_{j,1} = 0 \qquad \text{if } j > 1 \tag{1.2}$$

$$f_{1,i} \cdot f_{j,1} = \begin{cases} f_{1,1} & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases} \tag{1.3}$$

for all $i, j \in \{1, \ldots, n\}$, and

$$\sum_{i=1}^{n} f_{i,1} \cdot f_{1,i} = 1. \tag{1.4}$$

Note that $f_{1,1} f_{1,j} = f_{1,j}$ for all j because (1.1), (1.3), and (1.4) imply that

$$f_{1,1} f_{1,j} = f_{1,1}^2 f_{1,j} = \sum_{i=1}^{n} f_{i,1} f_{1,i} f_{1,j} - \sum_{i=2}^{n} f_{i,1} f_{1,i} f_{1,j} = f_{1,j} - \sum_{i=2}^{n} f_{i,1} \cdot 0 = f_{1,j}.$$

Likewise, we also have $f_{i,1} f_{1,1} = f_{i,1}$ for all i.

Let $\mathcal{M}_n$ be the algebra of $(n \times n)$-matrices with entries in F. We claim that $\mathcal{M}_n \cong M_n(F)$. For each $i, j \in \{1, \ldots, n\}$, let $e_{i,j} \in \mathcal{M}_n$ be the matrix with a 1 in the i-th row and j-th column and 0's elsewhere. Elements of the form $e_{i,j}$ are called *standard matrix units*, or simply *matrix units*; it is not difficult to verify that we can write any matrix in $\mathcal{M}_n$ as a linear combination of matrix units. Define a map $g \colon X \to \mathcal{M}_n$ where $g(f_{1,i}) = e_{1,i}$ and $g(f_{i,1}) = e_{i,1}$. Note that the multiplication in

$\mathcal{M}_n$ and its identity can be described by the relations

$$e_{i,j} \cdot e_{k,l} = \begin{cases} e_{i,l} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

and

$$1_{\mathcal{M}_n} = \sum_{i=1}^{n} e_{i,i} = \sum_{i=1}^{n} e_{i,1} \cdot e_{1,i}.$$

Some simple calculations imply that we can extend g to an algebra homomorphism $\varphi\colon M_n(F) \to \mathcal{M}_n$ by Theorem 1.8.

We claim that φ is, in fact, an isomorphism. In order to prove this, we shall first show that $B = \{f_{i,1}f_{1,j} \colon i,j \in \{1,\ldots,n\}\}$ forms a basis for $M_n(F)$. We begin by showing that B spans $M_n(F)$. First, note that relation (1.4) shows that multiples of the empty string are in the span of B. Since $f_{i,1}f_{1,1} = f_{i,1}$ for all i and $f_{1,1}f_{1,j} = f_{1,j}$ for all j, we also know that X is a subset of the span of B. Thus all products of length less than 2 are in span(B).

It now suffices to show that every product of length 3 and greater is equal to a product of length 0 or 2. Let $l > 2$ and let $p = x_1x_2\ldots x_l$ be a nonzero product consisting of elements of X; we claim that p is equal to a product of length 2. Let x_m be the left-most element in p of the form $f_{1,i}$ for some i. Relations (1.1) and (1.3) imply that $x_m \cdots x_l = f_{1,k}$ for some $k \in \{1,\ldots,n\}$ because $p \neq 0$. If $m = 1$, we are done because $p = f_{1,j} = f_{1,1}f_{1,j}$, so suppose $m > 1$. Since m is minimal, x_{m-1} must be of the form $f_{i,1}$ for some i. Relations (1.2) and (1.3) imply that $x_1 \ldots x_{m-1} = f_{j,1}$ for some $j \in \{1,\ldots,n\}$ because $p \neq 0$. Thus $p = f_{j,1}f_{1,k}$, proving the claim.

For linear independence, suppose there are $\lambda_{i,j} \in F$ for all $i,j \in \{1,\ldots,n\}$ such that

$$\sum_{i,j=1}^{n} \lambda_{i,j} f_{i,1} f_{1,j} = 0.$$

We claim that $\lambda_{i,j} = 0$ for all $i,j \in \{1,\ldots,n\}$. Using (1.3), we see that

$$f_{1,a}\left(\sum_{i,j=1}^{n} \lambda_{i,j} f_{i,1} f_{1,j}\right) f_{b,1} = \lambda_{a,b} f_{1,1} = 0$$

for all $a,b \in \{1,\ldots,n\}$. Since $f_{i,1}f_{1,1} = f_{i,1}$ and $f_{1,1}f_{1,j} = f_{1,j}$, we can only have $f_{1,1} = 0$ if

$f_{i,1} = f_{1,j} = 0$ for all i and j, which in turn implies $0 = 1$ by (1.4). This is, however, impossible in a field, so $\lambda_{a,b} = 0$ for all $a, b \in \{1, \ldots, n\}$. Therefore B is linearly independent and thus a basis for $M_n(F)$.

Since φ maps B bijectively onto the basis $\{e_{i,j} \colon i, j \in \{1, \ldots, n\}\}$ for $\mathcal{M}_n$, the claim that φ is an isomorphism now follows. We may thus use the definitions of $M_n(F)$ and $\mathcal{M}_n$ interchangeably.

1.2 Leavitt and Cuntz algebras

We now define the two algebras on which we shall focus our attention.

Definition 1.10. Let F be a field and let $n \in \mathbb{Z}_{>0}$. Define the n-th *Leavitt algebra* over F, denoted $L_n(F)$, as the universal unital associative algebra on generators

$$s_1, \ldots, s_n, t_1, \ldots, t_n$$

subject to the relations

$$t_i s_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for all $i, j \in \{1, \ldots, n\}$ and

$$\sum_{i=1}^{n} s_i t_i = 1.$$

Formally, the connection between the n-th Leavitt algebra $L_n(\mathbb{C})$ and the n-th Cuntz algebra $\mathcal{O}_n$ is topological, since $\mathcal{O}_n$ is meant to be the completion of the *-algebra equivalent of $L_n(\mathbb{C})$ when the latter algebra is equipped with an induced operator norm. We shall summarize the similarity of the two algebras for its importance in motivating the paper, but only briefly, as it strays from our intended trajectory.

We begin by defining the involution, or * operation:

Definition 1.11. A *-algebra (read as "star-algebra") over $\mathbb{C}$ is an algebra A over $\mathbb{C}$ with an additional operation $a \mapsto a^*$, called the *involution*, satisfying the following properties:

(1) $(a + b)^* = a^* + b^*$ for all $a, b \in A$;

(2) $(\lambda a)^* = \overline{\lambda} a^*$ for all $a \in A$ and $\lambda \in \mathbb{C}$;

(3) $(a^*)^* = a$ for all $a \in A$;

(4) $(ab)^* = b^* a^*$ for all $a, b \in A$.

One can refer to a^* as the *adjoint* of a.

A map that behaves like a homomorphism but reverses multiplication is called an *antihomomorphism*. Thus the star operation is an antihomomorphism on A. Given an algebra A, let the *opposite algebra* of A, denoted A^{op}, be identical to A but with reversed multiplication and conjugate scalar multiplication. An antihomomorphism $f \colon A \to B$ is then equivalent to a homomorphism $f \colon A \to B^{\mathrm{op}}$.

Example 1.12. We can make $L_n(\mathbb{C})$ a *-algebra by giving it an involution defined by $(s_i)^* = t_i$ and $(t_i)^* = s_i$ for all $i \in \{1, \ldots, n\}$.

In order to formally prove that the operation * satisfies Definition 1.11, let B be the opposite algebra of $L_n(\mathbb{C})$; that is, define B to be the algebra over $\mathbb{C}$ on generators

$$t_1, \ldots, t_n, s_1, \ldots, s_n,$$

satisfying relations

$$s_i \cdot t_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for all $i, j \in \{1, \ldots, n\}$ and

$$\sum_{i=1}^{n} t_i \cdot s_i = 1,$$

where vector multiplication is given by $a \cdot b = ba$ and scalar multiplication is given by $\lambda \cdot a = \overline{\lambda} a$. Thus B is $L_n(\mathbb{C})$, but with reversed vector multiplication and conjugate scalar multiplication.

Theorem 1.8 now gives a homomorphism $\sigma : L_n(\mathbb{C}) \to B$ such that $\sigma(s_i) = t_i$ and $\sigma(t_i) = s_i$ for all $i \in \{1, \ldots, n\}$. As we previously discussed, we may instead regard σ as an antihomomorphism from $L_n(\mathbb{C})$ to itself. Properties (1), (2), and (4) are consequences of σ being an antihomomorphism. For property (3), one need only observe that

$$\sigma^2(s_i) = \sigma(t_i) = s_i \qquad \text{and} \qquad \sigma^2(t_i) = \sigma(s_i) = t_i$$

for all $i \in \{1, \ldots, n\}$ in order to show that σ has order 2.

Therefore σ is a well-defined involution on $L_n(\mathbb{C})$. We can trivially extend this involution to $L_n(F)$ for an arbitrary non-complex field F by requiring that $\overline{k} = k$ for all $k \in F$. This will occasionally be used as shorthand so that we may denote t_i as s_i^*, although it has little mathematical utility outside being useful notation.

To construct the n-th Cuntz algebra $\mathcal{O}_n$, choose a nonzero Hilbert space H and a unital *-homomorphism $\pi\colon L_n(\mathbb{C}) \to L(H)$, where $L(H)$ is the algebra of bounded linear operators on H with the operator norm. Define a norm on $L_n(\mathbb{C})$ by setting $\|a\| = \|\pi(a)\|$ and define $\mathcal{O}_n$ to be the completion of $L_n(\mathbb{C})$ with respect to this norm. This definition does not depend on the choice of H or π, and makes $\mathcal{O}_n$ a normed $*$-algebra over $\mathbb{C}$ that satisfies the equation $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{O}_n$, and is complete with respect to its norm; such an algebra is called a C*-algebra. However, one may safely ignore the analytic properties of $\mathcal{O}_n$, as we prefer the following equivalent definition of a Cuntz algebra for its simplicity and similarity to Definition 1.10.

Definition 1.13. Let n be an integer greater than 1. Define the n-th *Cuntz algebra* $\mathcal{O}_n$ to be the universal unital C*-algebra on generators

$$s_1, \ldots, s_n$$

subject to the relations

$$s_i^* s_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{1.5}$$

for all $i, j \in \{1, \ldots, n\}$, and

$$\sum_{i=1}^{n} s_i s_i^* = 1. \tag{1.6}$$

The existence of universal C*-algebras is not as simple as the existence of universal algebras, as free C*-algebras do not exist. In particular, only certain types of algebraic relations produce universal C*-algebras; see [7] and [8] for more information.

As we most often interact with $\mathcal{O}_n$ as a subalgebra of a matrix algebra, we redefine the matrix algebra $M_n(\mathbb{C})$ as a C*-algebra.

Definition 1.14. Let n be a positive integer. Define $M_n(\mathbb{C})$ to be the universal unital associative C*-algebra generated by

$$g_1, g_2, \ldots, g_n$$

and subject to the relations

$$\begin{aligned} g_i g_j &= 0 \qquad \text{if } i > 1 \\ g_i g_j^* &= \begin{cases} g_1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \end{aligned}$$

for all $i, j \in \{1, \ldots, n\}$, and

$$\sum_{i=1}^{n} g_i^* g_i = 1.$$

This definition is a simplified form of the one provided in Example 1.9. The elements g_i and $f_{1,i}$ correspond with one another, as do g_i^* and $f_{i,1}$. The presence of the adjoint means that we can also combine relations (1.1) and (1.2) into a single relation. Although the above presentation of $M_n(\mathbb{C})$ is very slick, we prefer the notation $e_{i,j}$ over $g_i^* g_j$ as the the former notation is a bit shorter and the latter notation somewhat obscures the fact that it represents a matrix unit.

We could simplify the relations in Definition 1.14 slightly using properties of C*-algebras, but this would mangle things in terms of extending our results to the purely algebraic situation with $M_n(F)$ and $L_n(F)$.

The algebra we will be working over most often is the C*-algebra $M_m(\mathcal{O}_n)$, or $M_m(\mathbb{C}) \otimes \mathcal{O}_n$ for those familiar with tensor products. The elements of $M_m(\mathcal{O}_n)$ are the $(m \times m)$ matrices with entries in $\mathcal{O}_n$, and are written as sums of elements of the form $e_{i,j} \otimes v$, where $i, j \in \{0, \ldots, m\}$ and $v \in \mathcal{O}_n$. As one might expect, $e_{i,j} \otimes v$ denotes the matrix with v in the i-th row and j-th column and 0's everywhere else.

We formally define tensor products in the hopes that this demystifies the notation.

Definition 1.15. Let A and B be algebras over the same field F. Define the *tensor product* of A and B to be the algebra generated by elements of the form $a \otimes b$ and subject to the relations that, for any $a, a' \in A$, $b, b' \in B$, and $\lambda \in F$,

(i) $(a \otimes b) +_{A \otimes B} (a \otimes b') = a \otimes (b +_B b')$;

(ii) $(a \otimes b) +_{A \otimes B} (a' \otimes b) = (a +_A a' \otimes b)$;

(iii) $(\lambda a \otimes b) = (a \otimes \lambda b)$;

(iv) $(a \otimes b) \cdot_{A \otimes B} (a' \otimes b') = (a \cdot_A a' \otimes b \cdot_B b')$.

Addition and scalar multiplication take place to the left or to the right of $\otimes$, but not on both sides at once as in $A \times B$. On the other hand, multiplication is component-wise, just as in $A \times B$.

In the case of $M_m(\mathbb{C}) \otimes \mathcal{O}_n$, all this means is that $M_m(\mathbb{C}) \otimes \mathcal{O}_n$ is the universal unital associative C*-algebra on generators

$$e_{1,1} \otimes 1, e_{1,2} \otimes 1, \ldots, e_{1,m} \otimes 1, e_{1,1} \otimes s_1, e_{1,1} \otimes s_2, \ldots, e_{1,1} \otimes s_n$$

and subject to the relations

$$\begin{aligned}
(e_{1,i} \otimes 1)(e_{1,j} \otimes 1) &= 0 \qquad \text{if } i > 1 \\
(e_{1,i} \otimes 1)(e_{1,j} \otimes 1)^* &= \begin{cases} e_{1,1} \otimes 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \\
\sum_{i=1}^{m} (e_{1,i} \otimes 1)^*(e_{1,i} \otimes 1) &= 1_m \\
(e_{1,1} \otimes s_i)^*(e_{1,1} \otimes s_j)^* &= \begin{cases} e_{1,1} \otimes 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \\
\sum_{i=1}^{n} (e_{1,1} \otimes s_i)(e_{1,1} \otimes s_i)^* &= e_{1,1} \otimes 1.
\end{aligned}$$

Since the algebra $M_m(\mathbb{C})$ must be taken over the field $\mathbb{C}$ when we tensor it with $\mathcal{O}_n$, we omit the parentheses after M and write $M_m \otimes \mathcal{O}_n$ for convenience.

The only thing standing in the way of the problem statement is some elementary K-theory that we develop in the next section.

1.3 K_0 of Cuntz and Leavitt algebras

K-theory is a powerful tool which appears in a multitude of mathematical fields under several different guises. Of primary interest to us is Banach algebra K-theory, which has much in common with topological K-theory, although the two differ in important ways. There is another form of K-theory known as algebraic K-theory which we use for the Leavitt algebras. Although Banach algebra K-theory and algebraic K-theory are technically different, the K_0 groups of $\mathcal{O}_n$ and $L_n(F)$ are identical, and can be constructed in analogous ways. We shall focus on K_0 in the context of Banach algebra K-theory in this thesis.

The K-theory of C*-algebras (which are a special type of Banach algebra) consists of two covariant functors K_0 and K_1 from the category of C*-algebras to the category of abelian groups. For those unfamiliar with category theory, this means that each C*-algebra has two associated abelian groups $K_0(A)$ and $K_1(A)$, and every algebra homomorphism $f\colon A \to B$ induces group homomorphisms $f_*\colon K_i(A) \to K_i(B)$ for $i = 0, 1$. Additionally, $(\mathrm{id}_A)_* = \mathrm{id}_{K_i(A)}$ and $(f \circ g)_* = f_* \circ g_*$ when $f\colon B \to C$ and $g\colon A \to B$ are algebra homomorphisms. The functors K_0 and K_1 have a number of useful properties, although all we require is the fact that

$$K_0\big(M_n(\mathbb{C}) \otimes A\big) \cong K_0(A) \qquad \text{and} \qquad K_1\big(M_n(\mathbb{C}) \otimes A\big) \cong K_1(A)$$

for any positive integer n and any C*-algebra A. The action of tensoring a C*-algebra A by $M_n(\mathbb{C})$ is called *stabilization*, so we will refer to this property by noting that K-theory is unaffected by stabilization.

A thorough development of Banach algebra K-theory would take far too much time and significantly stray from the intended subject of this paper. Thus we shall limit our survey of Banach algebra K-theory strictly to what is necessary for understanding the project. As such, we take as known that

$$K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z} \qquad \text{and} \qquad K_1(\mathcal{O}_n) = 0$$

for all $n \geq 2$ ([5]). This gives us the K-theory of $M_m \otimes \mathcal{O}_n$ for all m and n, since K-theory is insensitive to stabilization.

Since K_1 is irrelevant to us, we now need only understand what it means for $K_0(\mathcal{O}_n)$ to be isomorphic to $\mathbb{Z}/(n-1)\mathbb{Z}$ in order to proceed to the problem statement and main results. The following overview is largely based on Joachim Cuntz' article "K-theory for Certain C^*-algebras," as well as notes provided to me by my advisor, Professor Phillips.

Definition 1.16. Let A be a unital *-algebra.

(1) We call an element $p \in A$ a *projection* if $p = p^* = p^2$. That is, p is both *self-adjoint* (its own adjoint) and an *idempotent* (every exponent of p is identical).

(2) We write that two projections $p, q \in A$ are *Murray-von Neumann equivalent*, denoted $p \sim q$, if there exists $s \in A$ such that $ss^* = p$ and $s^*s = q$.

(3) Two projections $p, q \in A$ are *orthogonal* if $pq = 0$. This is sometimes written as $p \perp q$.

(4) Given two projections $p, q \in A$, we write that q *dominates* p or $p \leq q$ if $qp = p$.

(5) A projection p is called *infinite* if there exists a projection $q \in A$ such that $q \sim p$, $q \leq p$, and $q \neq p$.

Note that, when two projections p and q are orthogonal, then $qp = (pq)^* = 0$ as well as $pq = 0$. Likewise, when p is dominated by q, then $pq = (qp)^* = p^* = p$. In the purely algebraic situation (without the involution), we will need to specify that two idempotents are orthogonal if both pq and qp are trivial; the extra structure the involution provides us means we can somewhat simplify these definitions.

Before proceeding further we verify that Murray-von Neumann equivalence is an equivalence relation and domination is a partial-ordering on projections.

Lemma 1.17. Let A be a unital *-algebra. Then Murray-von Neumann equivalence defines an equivalence relation on the projections of A.

Proof. Let $p, q, r \in A$ be projections. Clearly $p \sim p$ since $pp^* = p$ and $p^*p = p$. Thus $\sim$ is reflexive.

Suppose $p \sim q$; we claim that $q \sim p$. Then there exists $s \in A$ such that $ss^* = p$ and $s^*s = q$. By the definition of the involution, we also have $(s^*)(s^*)^* = s^*s = q$ and $(s^*)^*(s^*) = ss^* = p$. Hence $q \sim p$ and $\sim$ is symmetric.

Lastly, suppose that $p \sim q$ and $q \sim r$; we claim that $p \sim r$. Then there exist $s, t \in A$ such that $ss^* = p$, $s^*s = tt^* = q$, and $t^*t = r$. Since

$$(st)(st)^* = s(tt^*)s^* = s(s^*s)s^* = (ss^*)^2 = p$$

and

$$(st)^*(st) = t^*(s^*s)t = t^*(tt^*)t = (t^*t)^2 = r,$$

it follows that $p \sim r$. Hence $\sim$ is transitive and therefore an equivalence relation on A. □

Lemma 1.18. Let A be a unital *-algebra. Then domination defines a partial ordering on A the projections of A.

Proof. Let $p, q, r \in A$ all be projections. Clearly $p \leq p$, since $p^2 = p$. Thus $\leq$ is reflexive.

Suppose that $p \leq q$; we claim that $q \leq p$ implies that $p = q$. As we noted after Definition 1.16, $p \leq q$ implies that $qp = pq = p$ and $q \leq p$ implies that $pq = qp = q$. Combining these two equations shows that $p = q$, so $\leq$ is anti-symmetric.

Lastly, suppose that $p \leq q$ and $q \leq r$; we claim that $p \leq r$. Using the definition of domination, we find that

$$rp = r(qp) = (rq)p = qp = p.$$

Hence $p \leq r$ and $\leq$ is transitive. Therefore $\leq$ defines a partial ordering on A. □

Suppose we are given a C*-algebra A. We want to build $K_0(A)$ from the set of Murray-von Neumann equivalence classes of projections in A. However, there are issues with defining the addition of two equivalence classes $[p]$ and $[q]$. Ideally, we would want $[p] + [q]$ to be $[p + q]$, but $p + q$ is only a projection when $pq = -qp$. If this were true, then

$$pq = -qp = (-qp)p = (pq)p = p(qp) = p(-pq) = -pq,$$

so $pq = 0$ and $p \perp q$. Thus, if p and q are projections, $p + q$ is a projection if and only if p and q are orthogonal. If we could always choose $p_0 \sim p$ and $q_0 \sim q$ such that $p_0 \perp q_0$, then we could set $[p] + [q] = [p_0 + q_0]$.

This operation turns out to be well-defined, too. Suppose that, in addition to p_0 and q_0, there exist projections $p_1, q_1 \in A$ such that $p \sim p_1$, $q \sim q_1$, and $p_1 \perp q_1$. Then there exist $s, t \in A$ such that $ss^* = p_0$, $s^*s = p_1$, $tt^* = q_0$, and $t^*t = q_1$. Before we construct a Murray-von Neumann equivalence between $p_0 + q_0$ and $p_1 + q_1$, we must show that $s = p_0 s p_1$ and $t = q_0 t q_1$; the proofs for both identities are the same, so we only consider the case with s. Since $p_0 s p_1 = ss^* s p_1 = s p_1 = ss^* s$, it suffices to show that $s = ss^* s$. Note that

$$\begin{aligned}(ss^*s - s)^*(ss^*s - s) &= (s^*ss^* - s^*)(ss^*s - s)\\ &= s^*ss^*ss^*s - s^*ss^*s - s^*ss^*s + s^*s\\ &= p_1^3 - p_1^2 - p_1^2 + p_1 = 0.\end{aligned}$$

The C*-algebra identity $\|x^*x\| = \|x\|^2$ implies that $\|ss^*s - s\|^2 = 0$, so $ss^*s - s = 0$ by the definition of a norm. Consequently $st^* = (p_0 s p_1)(q_1 t^* q_0) = 0$ and $s^*t = (p_1 s^* p_0)(q_0 t q_1) = 0$ because $p_0 \perp q_0$ and $p_1 \perp q_1$. Hence

$$(s + t)(s + t)^* = ss^* + st^* + ts^* + tt^* = p_0 + q_0$$

and

$$(s + t)^*(s + t) = s^*s + s^*t + t^*s + t^*t = p_1 + q_1,$$

so $(p_0 + q_0) \sim (p_1 + q_1)$ and our operation is well-defined.

With this in mind, we define a family of projections over which this construction always works.

Definition 1.19. Let A be a C*-algebra and let $\mathcal{P} \subset A$ be a set of projections in A. We call $\mathcal{P}$ *properly infinite* if it satisfies the following axioms:

(A_1) If $p, q \in \mathcal{P}$ and $p \perp q$, then $p + q \in \mathcal{P}$.

(A_2) If $p \in \mathcal{P}$ and q is a projection in A such that $p \sim q$, then $q \in \mathcal{P}$.

(A_3) For all $p, q \in \mathcal{P}$, there exists $p_0 \in \mathcal{P}$ such that $p_0 \sim p$, $p_0 \leq q$, $p_0 \neq q$, and $q - p_0 \in \mathcal{P}$.

(A_4) If $q \in A$ is a projection such that $p \leq q$ for some $p \in \mathcal{P}$, then $q \in \mathcal{P}$.

The third axiom might look somewhat perplexing, but, given $p, q \in \mathcal{P}$, it allows us to choose $p_0 \sim p$ and $q_0 \sim q$ such that $p_0 \perp q_0$. First, choose $p_0 \in \mathcal{P}$ so that it satisfies (A_3) for $p, q \in \mathcal{P}$, and then choose q_0 so that it satisfies (A_3) for $q, q-p_0 \in \mathcal{P}$. Then

$$p_0q_0 = p_0(q-p_0)q_0 = (p_0q - p_0^2)q_0 = (p_0 - p_0)q_0 = 0.$$

We technically only need the first three axioms in order to make a group $G(\mathcal{P})$ from the Murray-von Neumann equivalence classes in $\mathcal{P}$, but require the fourth axiom so that $G(\mathcal{P})$ corresponds with the standard definition of $K_0(A)$.

Definition 1.20. Let A be a C*-algebra and let $\mathcal{P}$ be a properly infinite set of projections. Set $G(\mathcal{P}) = \{[p] \colon p \in \mathcal{P}\}$ and equip $G(\mathcal{P})$ with the binary operation

$$[p] + [q] = [p_0 + q_0],$$

where $p_0 \sim p$, $q_0 \sim q$, and $p_0 \perp q_0$.

Theorem 1.21. Let A be a C*-algebra and let $\mathcal{P}$ be a properly infinite set of projections. Then $G(\mathcal{P})$ is a well-defined abelian group.

The proof of Theorem 1.21 is a more detailed version of the proof of Theorem 1.4 in [5]. We require the following lemma, which we prove first.

Lemma 1.22. Let A be a C*-algebra, and let $d, e, f \in A$ be projections such that $e, f \leq d$, $e \sim f$, and $e \perp f$. Then $d - e \sim d - f$.

Proof. Choose $s \in A$ such that $ss^* = e$ and $s^*s = f$. Recall that $s = esf$. Thus $es = sf = s$ and $fs = se = 0$, the latter following from $e \perp f$. We also have $ds = sd = s$ because $e, f \leq d$. Therefore

$$s(d-e-f) = s - 0 - s = 0$$

and

$$(d-e-f)s = s - s - 0 = 0.$$

Taking the adjoint of both sides of the above equations shows that $(d-e-f)s^*$ and $s^*(d-e-f)$

both vanish. Hence

$$\begin{aligned}\big((d-e-f)+s\big)\big((d-e-f)+s\big)^* &= (d-e-f)^2+(d-e-f)s^*+s(d-e-f)+ss^*\\ &= (d-e-f)+0+0+f\\ &= d-e\end{aligned}$$

and

$$\begin{aligned}\big((d-e-f)+s\big)^*\big((d-e-f)+s\big) &= (d-e-f)^2+(d-e-f)s+s^*(d-e-f)+s^*s\\ &= (d-e-f)+0+0+e\\ &= d-f,\end{aligned}$$

so $d-e \sim d-f$. □

Proof of Theorem 1.21. We have already shown $+$ is well-defined in the discussion preceding Definition 1.19. Since $+$ is clearly commutative in $G(\mathcal{P})$, we now need only show $G(\mathcal{P})$ is a group.

We begin with associativity of $+$. Let $p, q, r \in \mathcal{P}$. Choose $p_0 \in \mathcal{P}$ such that $p_0 \sim p$, $p_0 \leq q$, and $q - p_0 \in \mathcal{P}$. Then choose $q_0 \in \mathcal{P}$ such that $q_0 \sim q$, $q_0 \leq q - p_0$, and $q - p_0 - q_0 \in \mathcal{P}$. Lastly, chose $r_0 \in \mathcal{P}$ such that $r_0 \sim r$ and $r_0 \leq q - p_0 - q_0$. Then

$$r_0p_0 = r_0(q-p_0-q_0)p_0 = r_0(p_0-p_0^2-0) = 0$$

and

$$r_0q_0 = r_0(q-p_0-q_0)q_0 = r_0(q_0-0-q_0^2) = 0,$$

so $r_0 \perp p_0$ and $r_0 \perp q_0$. Hence

$$\Big([p]+[q]\Big)+[r] = [p_0+q_0]+[r] = [p_0+q_0+r_0] = [p]+[q_0+r_0] = [p]+\Big([q]+[r]\Big),$$

so $+$ is associative.

We next show that $G(\mathcal{P})$ has an identity with respect to $+$. Let $p, q \in \mathcal{P}$. Using (A_3), choose $p_0 \in \mathcal{P}$ such that $p \sim p_0 \leq p$ and $p_0 \neq p$. We may assume $q \leq p_0$ without loss of generality by

(A$_3$); simply choose an element Murray-von Neumann equivalent to q which is dominated by p_0. Now use (A$_3$) to choose $q_0 \in \mathcal{P}$ such that $q \sim q_0 \leq q$ and $q_0 \neq q$. We claim that $[p-p_0] = [q-q_0]$, and that $[p-p_0]$ is the identity of $G(\mathcal{P})$.

Thus $(p-p_0)q = pp_0q - p_0q = q - q = 0$, so q and q_0 are orthogonal to $p - p_0$ (the latter is a consequence of q dominating q_0). Hence

$$[p-p_0] + [q-q_0] = [p - p_0 + q - q_0] = \big[p - (p_0 - q + q_0)\big].$$

We aim to show that the equivalence class on the right is equal to $[p-p_0]$.

Firstly, we claim that $p_0 - q + q_0 \sim p_0$. Choose $t \in A$ such that $tt^* = q$ and $t^*t = q_0$. Since $p_0 - q$ is orthogonal to both q and q_0, we have $(p_0 - q)t = (p_0 - q)t^* = 0$. Thus

$$\begin{aligned}\big((p_0-q)+t\big)\big((p_0-q)+t\big)^* &= (p_0-q)^2 + (p_0-q)t^* + t(p_0-q) + tt^* \\ &= (p_0 - q) + q_0 = p_0,\end{aligned}$$

and

$$\begin{aligned}\big((p_0-q)+t\big)^*\big((p_0-q)+t\big) &= (p_0-q)^2 + (p_0-q)t^* + t(p_0-q) + t^*t \\ &= (p_0 - q) + q_0 = p_0 - q + q_0,\end{aligned}$$

proving the claim.

By (A$_3$), there exists $p_1 \in \mathcal{P}$ such that $p \sim p_1 \leq p - p_0$. Thus $p_1 \sim p_0 \sim p_0 - q + q_0$ by transitivity of $\sim$. Furthermore,

$$p_1p_0 = p_1(p-p_0)p_0 = p_1(p_0 - p_0^2) = 0$$

and

$$p_1(p_0 - q + q_0) = p_1(p_0 - p_0q + p_0q_0) = 0,$$

so p_1 is orthogonal to both p_0 and $p_0 - q + q_0$. Therefore $p - p_0 \sim p - p_1 \sim p - (p_0 - q + q_0)$ by

Lemma 1.22, so

$$[p - p_0] + [q - q_0] = \big[p - (p_0 - q + q_0)\big] = [p - p_0].$$

An analogous argument proves that $[p - p_0] + [q - q_0] = [q - q_0]$, so $[p - p_0] = [q - q_0]$.

Now let $q \in \mathcal{P}$ and use (A$_3$) to choose $q_0 \in \mathcal{P}$ such that $q \sim q_0 \leq q$ and $q_0 \neq q$ (we no longer require $q \leq p_0$). That $[p - p_0]$ is the identity now follows from the fact

$$[q] + [p - p_0] = [q] + [q - q_0] = [q_0] + [q - q_0] = [q_0 + q - q_0] = [q].$$

Inverses also come easily now. Choose $q_1 \in \mathcal{P}$ such that $q \sim q_1 \leq q - q_0$. Then

$$[q] + [q - q_0 - q_1] = [q_1] + [q - q_0 - q_1] = [q_1 + q - q_0 - q_1] = [q - q_0],$$

so $[q - q_0 - q_1]$ is the inverse of $[q]$. $\square$

Cuntz showed that $G(\mathcal{P}) \cong K_0(A)$ when $\mathcal{P} \subset A$ is nonempty and properly infinite ([5], Theorem 1.4), so we shall simply define $K_0(A)$ to be $G(\mathcal{P})$ for a particular choice of $\mathcal{P}$.

Theorem 1.23. Let A be a simple C*-algebra and let $\mathcal{P}$ be the set of infinite projections in A. Then $\mathcal{P}$ is properly infinite.

Proof. We shall only verify that $\mathcal{P}$ satisfies (A$_1$), (A$_2$), and (A$_4$), as verifying (A$_3$) is significantly more involved. For a proof of (A$_3$), see the proof of Proposition 1.5 in [5]. Let $p \in \mathcal{P}$ and let $q \in A$ be a projection throughout. Choose $p_0 \in A$ such that $p \sim p_0 \leq p$ and $p \neq p_0$.

For (A$_1$), suppose that q is also infinite and that p and q are orthogonal. We claim that $p + q \sim p_0 + q \leq p + q$ and $p_0 + q \neq p + q$. Note that $p_0 q = p_0 p q = 0$, so $p_0 \perp q$. Thus $p + q \sim p_0 + q$ by the same reasoning we used to show $+$ is a well-defined operation on $G(\mathcal{P})$. We also have

$$(p + q)(p_0 + q) = pp_0 + pq + qp_0 + q^2 = p_0 + q,$$

so $p_0 + q \leq p + q$. Lastly, $p_0 + q = p + q$ contradicts our assumption that $p \neq p_0$, so $p_0 + q \neq p + q$. Therefore $p_0 + q_0 \neq p + q$ and (A$_1$) holds for $\mathcal{P}$.

For (A$_2$), suppose that $p \sim q$. Then there exists $s, t \in A$ such that $ss^* = p_0$, $s^*s = tt^* = p$, and

$t^*t = q$. Observe that

$$(t^*p_0t)^2 = t^*p_0pp_0t = t^*p_0t \qquad \text{and} \qquad (t^*p_0t)^* = t^*p_0^*t = t^*p_0t,$$

so t^*p_0t is a projection. We also have

$$t^*p_0tq = t^*p_0tt^*t = t^*p_0pt = t^*p_0t,$$

so $t^*p_0t \leq q$. Furthermore, $(p_0t)(p_0t)^* = p_0pp_0 = p_0$ and $(tp_0)^*(p_0t) = t^*p_0t$, so $t^*p_0t \sim\sim q$ by transitivity of $\sim$. Lastly, if $t^*p_0t = q$, then

$$p = p^2 = tt^*tt^* = tqt^* = t(t^*p_0t)t^* = pp_0p = p_0,$$

a contradiction, so $t^*p_0t \neq q$. Therefore q is infinite and (A_2) holds for $\mathcal{P}$.

For (A_4), suppose that $p \leq q$. Since $p \perp q - p$, we can express q as a sum of two orthogonal projections, $q - p$ and p. As $p_0 \sim p$, we have $q = q - p + p \sim q - p + p_0$. Multiplying q and $q - p + p_0$ yields

$$q(q - p + p_0) = q^2 - qp + qpp_0 = q - p + p_0,$$

so $q - p + p_0 \leq q$. Lastly, $q = q - p + p_0$ if and only if $p = p_0$, so $q \neq q - p + p_0$. Thus q is infinite and therefore $q \in \mathcal{P}$. $\square$

Definition 1.24. Let A be a simple unital C*-algebra and let $\mathcal{P} \subset A$ be the set of all infinite projections in A. Set $K_0(A) = G(\mathcal{P})$.

As it so happens, $\mathcal{O}_n$ is simple ([4], Theorem 1.12), so we can interpret $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ using the above definition. Furthermore, every projection in $\mathcal{O}_n$ is infinite, so we need not check whether a projection is infinite or not in the following discussion.

The projections we will be most interested in are those of the form $p_i = s_is_i^*$, or $p_{j,i} = s_jp_is_j^*$, or some longer self-adjoint string $p_{i_1,\ldots,i_k}$. Note that the projections $p_i = s_is_i^*$ are all orthogonal to one another and Murray-von Neumann equivalent to 1 by (1.5), as is the case for any $p_{i_1,\ldots,i_k}$. Thus

$[1] = [s_i s_i^*]$ for all i, and

$$[1] = \left[\sum_{i=1}^{n} s_i s_i^*\right] = \sum_{i=1}^{n}[s_i s_i^*] = n \cdot [1].$$

Hence $(n-1) \cdot [1] = 0$, so $[1 - s_i s_i^*] = 0$ for all $i \in \{1, \ldots, n\}$. We also have

$$[s_i s_i^*] + [1 - s_i s_i^* - s_i s_i s_i^* s_i^*] = [s_i s_i s_i^* s_i^* + 1 - s_i s_i^* - s_i s_i s_i^* s_i^*] = [1 - s_i s_i^*] = 0,$$

so the inverse of $s_i s_i^*$ is $[1 - s_i s_i^* - s_i s_i s_i^* s_i^*]$ for all i. Both of these results corroborate what we saw in the proof of Theorem 1.21. In [5], it is shown that $k \cdot [1] = 0$ if and only if $k \equiv 0 \bmod (n-1)$ and that every element in $K_0(\mathcal{O}_n)$ is a multiple of [1], which is how we know $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$.

On the other hand, the preferred generator of $K_0(M_m \otimes \mathcal{O}_n)$ is $[e_{1,1} \otimes 1]$, which we write as $[1]$ because $[e_{i,i} \otimes p] = [p]$ for all projections $p \in \mathcal{O}_n$. One should be aware of the following Murray-von Neumann equivalences on $M_m \otimes \mathcal{O}_n$, as they will appear often:

$$e_{1,1} \otimes 1 \sim e_{i,i} \otimes 1 \sim e_{i,i} \otimes s_j s_j^*.$$

Now that we have defined K_0, we can calculate the K_0-class of any projection in $M_m \otimes \mathcal{O}_n$ and are able to state the problem this thesis intends to (partially) answer.

For the purely algebraic case, there is an analogous derivation of $K_0(R)$ for purely infinite simple unital rings using idempotents instead of projections. One can reuse most of the proofs of the above statements with just a few changes in methodology. For example, two idempotents $e, f \in R$ are *algebraically Murray-von Neumann equivalent* if there exists $s, t \in R$ such that $st = e$ and $ts = f$. It does not immediately follow that $esf = s$ and $fte = t$ as in the C*-algebra case, but one can show that

$$(esf)(fte) = esfte = estste = e^4 = e$$

and

$$(fte)(esf) = ftesf = ftstsf = f^4 = f,$$

so one can choose $s_0, t_0 \in R$ such that $s_0 t_0 = e$, $t_0 s_0 = f$, $es_0 f = s_0$, and $ft_0 e = t_0$. Disregarding slight modifications like the one above, our definition of $K_0(R)$ for a purely infinite simple unital ring R is "isomorphic" to our definition of $K_0(A)$ for a purely infinite simple unital C*-algebra A.

By Theorem 4.1 in [2], we have

$$K_0\big(M_m(F) \otimes L_n(F)\big) \cong K_0(L_n(F)) \cong \mathbb{Z}/(n-1)\mathbb{Z}.$$

As far as the author knows, $K_1(L_n(F))$ is unknown, so one should ignore any occurrences of K_1 in the algebraic formulation of the problem using $L_n(F)$ instead of $\mathcal{O}_n$

2 The Automorphism Lifting Problem and Lifting Theorems

2.1 Problem statement

In its most general form, the problem we shall investigate is as follows.

Problem 2.1. Let n be an integer greater than 1 and let $K_*(A) = K_0(A) \oplus K_1(A)$ for a C*-algebra A. Given any element $\psi = (\psi_0, \psi_1)$ in the subgroup

$$\mathrm{Aut}(K_0(M_{n-1} \otimes \mathcal{O}_n)) \times \mathrm{Aut}(K_1(M_{n-1} \otimes \mathcal{O}_n)) \subset \mathrm{Aut}(K_*(M_{n-1} \otimes \mathcal{O}_n)),$$

write an explicit formula for an automorphism φ of $M_{n-1} \otimes \mathcal{O}_n$ such that the induced automorphism φ_* of $K_*(M_{n-1} \otimes \mathcal{O}_n)$ is equal to ψ.

Furthermore, if ψ has order l in $\mathrm{Aut}(K_*(M_{n-1} \otimes \mathcal{O}_n))$, can one choose φ so that it has order l in $\mathrm{Aut}(M_{n-1} \otimes \mathcal{O}_n)$?

The existence of solutions to all cases of Problem 2.1 are known (see [6]), but explicit formulas for φ are not, hence the phrasing of the problem.

There are three simplifications that can be made to the above problem, which is stated in such a way that it could be reused by replacing $M_{n-1} \otimes \mathcal{O}_n$ with any C*-algebra A. Firstly, $K_1(M_{n-1} \otimes \mathcal{O}_n) = 0$, so $\psi = (\psi_0, \psi_1) = (\psi_0, 0)$ and $K_*(M_{n-1} \otimes \mathcal{O}_n) = K_0(M_{n-1} \otimes \mathcal{O}_n)$. Secondly, as $K_0(M_{n-1} \otimes \mathcal{O}_n) = \mathbb{Z}/(n-1)\mathbb{Z}$, we have isomorphisms

$$\mathrm{Aut}\big(K_0(M_{n-1} \otimes \mathcal{O}_n)\big) \cong \mathrm{Aut}\big(\mathbb{Z}/(n-1)\mathbb{Z}\big) \cong \big(\mathbb{Z}/(n-1)\mathbb{Z}\big)^{\times},$$

where $R^\times$ is the unit subgroup of a ring R. The second isomorphism is derived by noting that each endomorphism ψ of $(\mathbb{Z}/(n-1)\mathbb{Z}, +)$ is determined by $\psi(1)$, and that ψ is an isomorphism if and only if $\psi(1)$ generates $(\mathbb{Z}/(n-1)\mathbb{Z}, +)$. Thus ψ is an automorphism if and only if $\psi(1)$ is relatively prime with $n-1$. We can thus narrow down the choice of an automorphism $\psi = (\psi_0, 0)$ to choosing an integer $k \in \{1, \ldots, n-1\}$ satisfying $\gcd(k, n-1) = 1$. Lastly, an automorphism ψ of $(\mathbb{Z}/(n-1)\mathbb{Z}, +)$ given by $1 \mapsto k$ has the same order as $k \in (\mathbb{Z}/(n-1)\mathbb{Z}, \cdot)$ because $\psi^l(1) = k^l$. The existence of a minimal positive integer l such that $k^l \equiv 0 \bmod (n-1)$ is guaranteed because $(\mathbb{Z}/(n-1)\mathbb{Z})^\times$ is finite.

We can therefore restate the problem like so.

Problem 2.2. Let n be an integer greater than 1. For each $k \in \{1, \ldots, n-1\}$ relatively prime to $n-1$, write an explicit formula for an automorphism φ of $M_{n-1} \otimes \mathcal{O}_n$ such that $\varphi_*([1]) = k \cdot [1]$.

Let l be the multiplicative order of k in $\mathbb{Z}/(n-1)\mathbb{Z}$. Can one choose φ so that l is also the minimal positive integer satisfying $\varphi^l = \mathrm{id}_{M_{n-1} \otimes \mathcal{O}_n}$?

Before moving on to the main results, the author feels the need to explain why $M_{n-1} \otimes \mathcal{O}_n$ was chosen instead of $M_m \otimes \mathcal{O}_n$ for some $m \neq n-1$, or simply $\mathcal{O}_n$. As one will recall from our overview of K-theory, the Murray-von Neumann equivalence class of $1 \in \mathcal{O}_n$ generates $K_0(\mathcal{O}_n)$. Since isomorphisms send identities to identities, any automorphism of $\mathcal{O}_n$ sends 1 to 1, so its induced map on K_0 is the identity and thus trivial.

The choice to work over $M_{n-1} \otimes \mathcal{O}_n$ instead of some other matrix algebra $M_m \otimes \mathcal{O}_n$ is somewhat more arbitrary, but $M_{n-1} \otimes \mathcal{O}_n$ has the advantage that the K_0-class of its identity 1_{n-1} is 0:

$$[1_{n-1}] = \left[\sum_{i=1}^{n-1} e_{i,i} \otimes 1 \right] = \sum_{i=1}^{n-1} [e_{i,i} \otimes 1] = (n-1) \cdot [1] = 0.$$

More generally, the K_0-class of the identity $1_m \in M_m \otimes \mathcal{O}_n$ is $(m \bmod (n-1)) \cdot [1]$. Say $m \equiv m' \bmod (n-1)$ for some $m' \in \{0, \ldots, n-1\}$. Then every automorphism φ of $M_m \otimes \mathcal{O}_n$ satisfies $\varphi_*(m' \cdot [1]) = m' \cdot [1]$, so we can only consider automorphisms of $\mathbb{Z}/(n-1)\mathbb{Z}$ that fix m'. If $m' \neq 0$, this either limits our choices for automorphism of $\mathbb{Z}/(n-1)\mathbb{Z}$, or trivializes the entire problem by forcing $\psi(1) = 1$. For example, the only automorphism of $\mathbb{Z}/4\mathbb{Z}$ that sends 3 to itself is the identity, so Problem 2.1 is trivial for $M_3 \otimes \mathcal{O}_5$. However, every automorphism of $\mathbb{Z}/4\mathbb{Z}$ sends 2 to itself, so

Problem 2.1 is completely reasonable for $M_2 \otimes \mathcal{O}_5$. Regardless, we decide to focus on $M_{n-1} \otimes \mathcal{O}_n$ so as to allow the maximum number of choices of $\psi\colon K_0(M_{n-1} \otimes \mathcal{O}_n) \to K_0(M_{n-1} \otimes \mathcal{O}_n)$. Along the way, we happen to obtain partial answers to the versions of Problem 2.2 for very specific choices of $M_m \otimes \mathcal{O}_n$.

Finally, some words should be said about why one would want to answer these problems. A solution to the first part of Problem 2.2 would yield a function $\sigma\colon \text{Aut}(K_0(M_{n-1} \otimes \mathcal{O}_n)) \to \text{Aut}(M_{n-1} \otimes \mathcal{O}_n)$ such that $\sigma(\psi)_* = \psi$ for all $\psi \in \text{Aut}(K_0(M_{n-1} \otimes \mathcal{O}_n))$. This directly shows that the K-theoretic map $\text{Aut}(M_{n-1} \otimes \mathcal{O}_n) \to \text{Aut}(K_0(M_{n-1} \otimes \mathcal{O}_n))$ is surjective by constructing an element in the preimage of each element in $\text{Aut}(K_0(M_{n-1} \otimes \mathcal{O}_n))$. If one could define σ so that $\sigma(\psi) \circ \sigma(\psi') = \sigma(\psi') \circ \sigma(\psi)$ for all ψ and ψ' and $\sigma(\psi)$ has the same order as ψ for all ψ, then σ would be a homomorphism. In particular, a full solution to Problem 2.2 yields a homomorphism σ when $\text{Aut}(K_0(M_{n-1} \otimes \mathcal{O}_n))$ is cyclic. In this case, one could regard σ as a group action of $\text{Aut}(K_0(M_{n-1} \otimes \mathcal{O}_n))$ on $M_{n-1} \otimes \mathcal{O}_n$. Both partial and full answers reveal valuable information about the algebraic invariant $K_0(M_{n-1} \otimes \mathcal{O}_n)$ and how its automorphism group relates to $\text{Aut}(M_{n-1} \otimes \mathcal{O}_n)$. These solutions will also apply to the purely algebraic case $M_{n-1}(F) \otimes L_n(F)$, for which the answer to Problem 2.1 is not known.

Algebraic invariants of C*-algebras, especially those related to the functors K_0 and K_1, play a major role in the classification of C*-algebras. Various sorts of equivalences between C*-algebras imply or can be deduced from isomorphic K-theories and K-theoretic properties. This is the main motivation for solving Problem 2.2, which is the simplest case of a broader problem dealing with a family of C*-algebras that contains the Cuntz algebra $\mathcal{O}_n$ for all n.

2.2 Ignoring the order requirement

Here we address the first part of Problem 2.2 and ignore the second part, which asks that the lifted automorphism has the same finite order as the K_0-automorphism. As it so happens, relaxing this requirement makes it possible to construct lifts of every automorphism of $K_0(M_{n-1} \otimes \mathcal{O}_n)$.

Theorem 2.3. Let n and k be integers such that $1 < k < n$ and $\gcd(k, n-1) = 1$. Then there exists an automorphism $\varphi\colon M_{n-1} \otimes \mathcal{O}_n \to M_{n-1} \otimes \mathcal{O}_n$ such that $\varphi_*([1]) = k \cdot [1]$.

This is, in fact, a corollary of a more specific result:

Theorem 2.4. Let n and k be integers such that $1 < k < n$ and $\gcd(k, n-1) = 1$, and let l be the multiplicative order of k in $\mathbb{Z}/(n-1)\mathbb{Z}$. Set $m = \sum_{i=0}^{l-1} k^i$. Then there exists an automorphism $\varphi\colon M_m \otimes \mathcal{O}_n \to M_m \otimes \mathcal{O}_n$ such that $\varphi_*([1]) = k \cdot [1]$.

We can derive Theorem 2.3 from Theorem 2.4 by observing that

$$(k-1)m = k^l - 1 \equiv 0 \bmod (n-1),$$

with n, k, l, and m as in Theorem 2.4. By Theorem 5.2 of [1], there exists an isomorphism $\tau\colon M_{(k-1)m} \otimes \mathcal{O}_n \to M_{n-1} \otimes \mathcal{O}_n$ that preserves K_0-classes. Then $\tau \circ (\mathrm{id}_{M_{k-1}} \otimes \varphi) \circ \tau^{-1}$ is an automorphism satisfying the conditions of Theorem 2.3, where φ is as in Theorem 2.4. Moreover, $\tau \circ (\mathrm{id}_{M_{k-1}} \otimes \varphi) \circ \tau^{-1}$ has the same order as φ, so this method is applicable when we do finally consider finite order.

Integers like m happen to be crucial in the proof of Theorem 2.4, so we formally define an integer function that generalizes its construction for convenience.

Notation 2.5. Let $f\colon (\mathbb{Z}_{\geq 0})^2 \to \mathbb{Z}_{\geq 0}$ be the function defined by

$$f(k,p) = \sum_{i=0}^{p-1} k^i$$

for all $k, p \in \mathbb{Z}_{\geq 0}$.

Each exponent k^i represents a $(k^i \times k^i)$ matrix in $M_{f(k,l)} \otimes \mathcal{O}_n$. We plan to define φ on $M_{f(k,l)} \otimes \mathcal{O}_n$ by the following family of isomorphisms.

Lemma 2.6. Let n and k be integers such that $1 < k < n$ and $\gcd(k, n-1) = 1$, and let l be the multiplicative order of k in $\mathbb{Z}/(n-1)\mathbb{Z}$. For all $p \in \{0, \ldots, l-1\}$, identify the matrix algebra $M_{k^p}(\mathcal{O}_n)$ with the upper left $(k^p \times k^p)$ matrix in $M_{k^{l-1}}(\mathcal{O}_n)$.

Then there exists a family of isomorphisms $\psi_p\colon M_{k^p} \otimes \mathcal{O}_n \to M_{k^{p+1}} \otimes \mathcal{O}_n$ for all $p \in \{0, \ldots, l-2\}$, such that ψ_{p+1} restricts to ψ_p on $M_{k^p}(\mathcal{O}_n)$ for all $p \in \{0, \ldots, l-2\}$.

Proof. Once we have defined $\psi_0\colon M_1 \otimes \mathcal{O}_n \to M_k \otimes \mathcal{O}_n$, we will set $\psi_i = \mathrm{id}_{M_{k^p}} \otimes \psi_0$ for all $i \in \{2, \ldots, l-2\}$. For the student unfamiliar with the tensor product of two maps, note that

$M_{k^{p+1}} \otimes \mathcal{O}_n \cong M_{k^p} \otimes M_k \otimes \mathcal{O}_n$ can be decomposed into k^{2p} square matrices of dimension k. Then ψ_i can be defined so that

$$\psi_i(e_{i,j} \otimes v) = e_{i,j} \otimes \psi_0(v),$$

where $e_{i,j}$ identifies the $(k \times k)$ matrix in $M_{k^{p+1}} \otimes \mathcal{O}_n$ and $\psi_0(v)$ identifies what goes inside the $(k \times k)$ matrix.

The existence of ψ_0 is a consequence of a result of Abrams, Ánh, and Pardo, who showed that $\mathcal{O}_n \cong M_d \otimes \mathcal{O}_n$ if and only if $\gcd(d, n-1) = 1$ ([1], Theorem 4.14). Moreover, one can provide an explicit formula for each of these isomorphisms, which we omit for brevity. In our case, we have $\gcd(k, n-1) = 1$ by assumption, so let ψ_0 be as defined in [1], save for the fact we tensor $\mathcal{O}_n$ with the trivial matrix algebra M_1.

It now immediately follows from the definition of each ψ_i that $\psi_{p+1}|_{M_{k^p} \otimes \mathcal{O}_n} = \psi_p$. $\square$

As $\psi_p(e_{1,1} \otimes 1) = \psi_0(e_{1,1} \otimes 1) = 1_k$ for all p, we know that $(\psi_p)_*([1]) = k \cdot [1]$ because the K_0-class of an element in $M_{k^p} \otimes \mathcal{O}_n$ is identical to its K_0-class in $M_{f(k,l)} \otimes \mathcal{O}_n$. This will be important in the following proof.

Proof of Theorem 2.4. Unlike in Lemma 2.6, here we identify the matrix algebras

$$M_{k^0} \otimes \mathcal{O}_n, M_{k^1} \otimes \mathcal{O}_n, \ldots, M_{k^{l-1}} \otimes \mathcal{O}_n$$

with matrices along the diagonal of $M_{f(k,l)} \otimes \mathcal{O}_n$. In particular, for all $p \in \{0, \ldots, l-1\}$, we identify the matrix algebra $M_{k^p} \otimes \mathcal{O}_n$ with the $(k^p \times k^p)$ matrix in $M_{f(k,l)} \otimes \mathcal{O}_n$ whose top left corner is located in the $(1 + f(k,p))$-th row and $(1 + f(k,p))$-th column of $M_{f(k,l)} \otimes \mathcal{O}_n$. Thus $e_{i,j} \otimes 1 \in M_{k^p} \otimes \mathcal{O}_n$ if and only if

$$1 + f(k,p) \leq i, j \leq f(k, p+1).$$

The intention of this identification is to have φ map $M_{k^p} \otimes \mathcal{O}_n$ onto $M_{k^{p+1}} \otimes \mathcal{O}_n$ for all $p \in \{0, \ldots, l-2\}$ and to map $M_{k^{l-1}} \otimes \mathcal{O}_n$ onto $M_{k^0} \otimes \mathcal{O}_n$. We shall achieve this by using the isomorphisms from Lemma 2.6.

Let $\psi_0, \ldots, \psi_{l-2}$ be as in Lemma 2.6 and define $\psi_{l-1}\colon M_{k^{l-1}} \otimes \mathcal{O}_n \to M_1 \otimes \mathcal{O}_n$ by

$$\psi_{l-1} = \psi_0^{-1} \circ \psi_1^{-1} \circ \cdots \circ \psi_{l-2}^{-1}.$$

We begin by defining projections $D_1, \ldots, D_{n-1}$ that describe where φ will send the diagonal elements $e_{i,i} \otimes 1$ of $M_{f(k,l)} \otimes \mathcal{O}_n$. Let $i \in \{1, \ldots, f(k,l)\}$. Then there exists $p \in \{1, \ldots, l-1\}$ such that $f(k,p) + 1 \leq i \leq f(k,p+1)$. Set

$$D_i = \psi_p(e_{i,i} \otimes 1).$$

We want $\varphi(e_{i,i} \otimes 1) = D_i$, so we must define $\varphi(e_{1,i} \otimes 1)$ so that $D_1 = \varphi(e_{1,i} \otimes 1)\varphi(e_{1,i} \otimes 1)^*$ and $D_i = \varphi(e_{1,i} \otimes 1)^* \varphi(e_{1,i} \otimes 1)$. Thus we must construct Murray-von Neumann equivalences between D_1 and D_i for all $i \in \{1, \ldots, f(k,l)\}$. For $i \in \{1, \ldots, f(k,l-1)\}$, this is straightforward; the D_i in this range can be written as

$$D_i = \sum_{\alpha=1}^{k} e_{k(i-1)+\alpha+1, k(i-1)+\alpha+1} \otimes 1.$$

Set

$$U_i = \sum_{\alpha=1}^{k} e_{\alpha+1, k(i-1)+\alpha+1} \otimes 1$$

for all $i \in \{1, \ldots, f(k,l-1)\}$. Then $U_i U_i^* = D_1$ and $U_i^* U_i = D_i$ for all $i \in \{1, \ldots, f(k,l-1)\}$. Note that $U_1 = D_1$.

Showing that $[D_i] = k \cdot [1]$ in $K_0(M_1 \otimes \mathcal{O}_n)$ for all $i \in \{f(k,l-1)+1, \ldots, f(k,l)\}$ is slightly less straightforward. Recall that $(\psi_p)_*\colon K_0(M_{k^p} \otimes \mathcal{O}_n) \to K_0(M^{k^{p+1}} \otimes \mathcal{O}_n)$ is given by $[1] \mapsto k \cdot [1]$ for all p. Since k has order l in $\mathbb{Z}/(n-1)\mathbb{Z}$, the inverse map $(\psi_p^{-1})_*$ is given by $[1] \mapsto k^{l-1} \cdot [1]$ for all $p \in \{0, \ldots, l-2\}$. Functoriality of K_0 implies that

$$(\psi_{l-1})_* = (\psi_0^{-1})_* \circ (\psi_1^{-1})_* \circ \cdots \circ (\psi_{l-2}^{-1})_*$$

and therefore that $(\psi_{l-1})_*([1]) = (k^{l-1})^{l-1} \cdot [1]$. As

$$(k^{l-1})^{l-1} = k^{l^2-2l+1} = k^{l^2-2l} \cdot k \equiv 1 \bmod (n-1),$$

it follows that $D_i = \psi_{l-1}(e_{i,i} \otimes 1)$ has K_0-class $k \cdot [1]$ for all $i \in \{f(k, l-1), \ldots, f(k, l)\}$.

We claim that any projection whose K_0-class is $k \cdot [1] \in K_0(\mathcal{O}_n)$ can be written as a sum of k orthogonal projections, each of which is Murray-von Neumann equivalent to 1. This happens to be true of any purely infinite simple unital C*-algebra, as then (A$_3$) holds for all nonzero projections (because every projection is infinite) and any two nonzero projections are Murray-von Neumann equivalent if and only if they have the same K_0-class. Let $p \in \mathcal{O}_n$ be a projection and let $p = k \cdot [1]$. If $k = 1$, we are done, so suppose $k > 1$. Then there is a projection $q_1 \in \mathcal{O}_n$ such that $1 \sim q_1 \leq p$ and $q_1 \neq p$. Therefore $p = (p - q_1) + q_1$ and $p - q_1 \perp q_1$, so

$$[p - q_1] + [1] = [p - q_1] + [q_1] = [p - q_1 + q_1] = [p] = k \cdot [1].$$

Consequently we have $[p - q_1] = (k-1) \cdot [1]$, and we can repeat this process for $p - q_1$ until we have projections $q_1 \sim \cdots \sim q_{k-1} \sim 1$ such that $q_j \leq p - \sum_{i=1}^{j-1} q_i$ for all j. Set $q_k = p - \sum_{i=1}^{k-1} q_i$. Then $p = \sum_{i=1}^{k} q_i$ and $q_i \perp q_{i+1}$ for all $i < k-1$, so $q_i \perp q_j$ for all i, j. This proves the claim.

Thus, for each $i \in \{f(k, l-1) + 1, \ldots, f(k, l)\}$, there exists $w_{i,1}, w_{i,2}, \ldots, w_{i,k} \in \mathcal{O}_n$ such that

$$D_i = e_{1,1} \otimes \sum_{\alpha=1}^{k} w_{i,\alpha}^* w_{i,\alpha},$$

$w_{i,\alpha} w_{i,\alpha}^* = 1$ for all i and α, and $w_{i,\alpha}^* w_{i,\alpha} \perp w_{i,\beta}^* w_{i,\beta}$ if and only if $\alpha \neq \beta$. Thus $w_{i,\alpha} w_{i,\beta}^* = 0$ if and only if $\alpha \neq \beta$. Set

$$U_i = \sum_{\alpha=1}^{k} e_{\alpha+1,1} \otimes w_{\alpha,i}$$

for all $i \in \{f(k, l-1) + 1, \ldots, f(k, l)\}$. These U_i also satisfy $U_i U_i^* = D_1$ and $U_i^* U_i = D_i$.

For all $j \in \{1, \ldots, n\}$, set

$$V_j = \psi_0(e_{1,1} \otimes s_j).$$

We claim that there exists a unique endomorphism $\varphi \colon M_{f(k,l)} \otimes \mathcal{O}_n \to M_{f(k,l)} \otimes \mathcal{O}_n$ such that $\varphi(e_{1,i} \otimes 1) = U_i$ for all $i \in \{1, \ldots, f(k, l)\}$ and $\varphi(e_{1,1} \otimes s_j) = V_j$ for all $j \in \{1, \ldots, n\}$. In order to prove this, we shall show that the following relations hold for all appropriate values of i and j:

$$U_iU_j = 0 \qquad \text{if } i > 1 \tag{1}$$

$$U_iU_j^* = \begin{cases} U_1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{2}$$

$$\sum_{i=1}^{n-1} U_i^*U_i = 1_{f(k,l)} \tag{3}$$

$$V_i^*V_j = \begin{cases} U_1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{4}$$

$$\sum_{j=1}^{n} V_jV_j^* = U_1. \tag{5}$$

We begin by proving relation (1). Let $i, j \in \{1, \ldots, f(k,l)\}$ and suppose $i > 1$. If $i \leq f(k, l-1)$, then

$$U_i = \sum_{\alpha=1}^{k} e_{\alpha+1,k(i-1)+\alpha+1} \otimes 1$$

and $k(i-1) + \alpha + 1 > \beta + 1$ for all $\alpha, \beta \in \{1, \ldots, k\}$. Alternatively, if $i > f(k, l-1)$, then

$$U_i = \sum_{\alpha=1}^{k} e_{\alpha+1,1} \otimes w_{\alpha,i}$$

and $1 < \beta + 1$ for all $\beta \in \{1, \ldots, k\}$. Thus

$$U_i \sum_{\beta=1}^{k} e_{\beta+1,k(j-1)+\beta+1} \otimes 1 = 0$$

and

$$U_i \sum_{\beta=1}^{k} e_{\beta+1,1} \otimes w_{\beta,j} = 0$$

by definition of matrix unit multiplication. Therefore $U_iU_j = 0$, proving relation (1).

We move on to proving relation (2). We have already shown that $U_iU_i^* = D_1 = U_1$, so we need only check the case $i \neq j$, and specifically the case when $i < j$ since $U_iU_j^* = 0$ implies that $U_jU_i^* = (U_iU_j^*)^* = 0$. Let $i, j \in \{1, \ldots, f(k,l)\}$. Then there exists $p, p' \in \{0, \ldots, l-2\}$ such that $1 + f(k,p) \leq i \leq f(k,p+1)$ and $1 + f(k,p') \leq j \leq f(k,p'+1)$. There are three cases to consider depending on where i and j are in relation to $f(k,l-1)$.

Case 1: $1 \leq i < j \leq f(k,l-1)$.

Since $i < j$, we have $k(j-i) \geq k$. Thus

$$k(i-1) + \alpha + 1 \neq k(j-1) + \beta + 1$$

for all $\alpha, \beta \in \{1, \ldots, k\}$, so

$$U_i U_j^* = \left(\sum_{\alpha=1}^{k} e_{\alpha+1,k(i-1)+\alpha+1} \otimes 1 \right) \left(\sum_{\beta=1}^{k} e_{k(i-1)+\beta+1,\beta+1} \otimes 1 \right) = 0.$$

Case 2: $1 \leq i \leq f(k,l-1) < j \leq f(k,l)$.

Clearly $k(i-1) + \alpha + 1 > 1$ for all $\alpha \in \{1, \ldots, k\}$, so

$$U_i U_j^* = \left(\sum_{\alpha=1}^{k} e_{\alpha+1,k(i-1)+\alpha+1} \otimes 1 \right) \left(\sum_{\beta=1}^{k} e_{1,\beta+1} \otimes w_{j,\beta}^* \right) = 0.$$

Case 3: $f(k,l-1) < i < j \leq f(k,l)$

Since $e_{i,i} \otimes 1$ and $e_{j,j} \otimes 1$ are orthogonal projections in $M_{p^{l-1}} \otimes \mathcal{O}_n$ and the map ψ_{l-1} is an isomorphism, the projections

$$\psi_{l-1}(e_{i,i} \otimes 1) = D_i = U_i^* U_i$$

and

$$\psi_{l-1}(e_{j,j} \otimes 1) = D_j = U_j^* U_j$$

are also orthogonal. The identity

$$(U_i^* U_i)(U_j^* U_j) = D_i D_j = 0$$

implies that $U_i U_j^* = 0$, proving relation (2) holds.

For relation (3), we use the fact that, for all $i \in \{1, \ldots, f(k,l)\}$,

$$U_i U_i^* = D_i = \psi_p(e_{i,i} \otimes 1)$$

for the unique $p \in \{0, \ldots, l-2\}$ satisfying $f(k,p)+1 \le i \le f(k,p+1)$. Since

$$\sum_{i=1+f(k,p)}^{f(k,p+1)} \psi_p(e_{i,i} \otimes 1) = \psi_p(1_{k^p}) = 1_{k^{p+1}} = \sum_{i=1+f(k,p+1)}^{f(k,p+2)} e_{i,i} \otimes 1$$

for all $p < l-1$, we have

$$\begin{aligned}
\sum_{i=1}^{f(k,l)} U_i^* U_i &= \sum_{p=0}^{l-1} \sum_{i=f(k,p)+1}^{f(k,p+1)} D_i \\
&= \sum_{p=0}^{l-2} \sum_{i=f(k,p)+1}^{f(k,p+1)} \Big(\psi_p(e_{i,i} \otimes 1)\Big) + \sum_{i=f(k,l-1)+1}^{f(k,l)} \psi_{l-1}(e_{i,i} \otimes 1) \\
&= \sum_{p=0}^{l-2} 1_{k^{p+1}} + \psi_{l-1}\left(\sum_{i=f(k,l-1)+1}^{f(k,l)} e_{i,i} \otimes 1 \right). \qquad (\star)
\end{aligned}$$

As $\psi_{l-1} \colon M_{k^{l-1}} \otimes \mathcal{O}_n \to M_1 \otimes \mathcal{O}_n$ is also an isomorphism, we have

$$\psi_{l-1}\left(\sum_{i=f(k,l-1)+1}^{f(k,l)} e_{i,i} \otimes 1 \right) = \psi_{l-1}(1_{k^{l-1}}) = 1_{k^0} = e_{1,1} \otimes 1.$$

Thus, continuing from $(\star)$,

$$\begin{aligned}
\sum_{i=1}^{f(k,l)} U_i^* U_i &= \sum_{p=0}^{l-2} 1_{k^{p+1}} + \psi_{l-1}\left(\sum_{i=f(k,l-1)+1}^{f(k,l)} e_{i,i} \otimes 1 \right) \\
&= \sum_{p=0}^{l-2} \sum_{i=f(k,p)+1}^{f(k,p+1)} \Big(e_{i,i} \otimes 1\Big) + e_{1,1} \otimes 1 \\
&= \sum_{i=f(k,1)+1}^{f(k,l)} \Big(e_{i,i} \otimes 1\Big) + e_{1,1} \otimes 1 \\
&= 1_{f(k,l)}.
\end{aligned}$$

Relations (4) and (5) follow directly from $\psi_0 \colon M_1 \otimes \mathcal{O}_n \to M_k \otimes \mathcal{O}_n$ being an isomorphism and the defining relations of $\mathcal{O}_n$, but we shall check both to be thorough. Let $i, j \in \{1, \ldots, n\}$.

Then

$$V_i^* V_j = \psi_0(e_{1,1} \otimes s_i^*)\psi_0(e_{1,1} \otimes s_j) = \psi_0(e_{1,1} \otimes s_i^* s_j) = \begin{cases} \psi_0(e_{1,1} \otimes 1) & \text{if } i = j \\ \psi_0(0) & \text{if } i \neq j. \end{cases}$$

Since $\psi_0(e_{1,1} \otimes 1) = D_1 = U_1$, the elements $V_1, \ldots, V_n$ satisfy relation (4).

Lastly, relation (5) follows from the calculation

$$\sum_{i=1}^{n} V_i V_i^* = \sum_{i=1}^{n} \psi_0(e_{1,1} \otimes s_i)\psi_0(e_{1,1} \otimes s_i)^* = \psi_0\left(e_{1,1} \otimes \sum_{i=1}^{n} s_i s_i^*\right) = \psi_0(e_{1,1} \otimes 1) = U_1.$$

Therefore there exists a unique endomorphism φ of $M_{n-1} \otimes \mathcal{O}_n$ such that $\varphi(e_{1,i} \otimes 1) = U_i$ for all $i \in \{1, \ldots, f(k,l)\}$ and $\varphi(e_{1,1} \otimes s_j) = V_j$ for all $j \in \{1, \ldots, n\}$. We claim that φ is an automorphism. Given that φ is nonzero and the matrix algebra $M_{f(k,l)} \otimes \mathcal{O}_n$ is simple, we know that φ is injective. In order to show that φ is surjective and thus bijective, we need only show that the generating set

$$\left\{e_{1,i} \otimes 1 \colon i \in \{1, \ldots, f(k,l)\}\right\} \cup \left\{e_{1,1} \otimes s_j \colon j \in \{1, \ldots, n\}\right\}$$

is contained within the image of φ.

We begin by showing $e_{1,i} \otimes 1 \in \text{Im}(\varphi)$ for all $i \in \{1, \ldots, n-1\}$. This is easily shown once we know that $e_{1,2} \otimes 1$ is in the image of φ. Recall that

$$\sum_{i=f(k,l-1)+1}^{f(k,l)} U_i^* U_i = \sum_{i=f(k,l-1)+1}^{f(k,l)} \sum_{\alpha=1}^{k} e_{1,1} \otimes w_{i,\alpha}^* w_{i,\alpha} = e_{1,1} \otimes 1,$$

so

$$\sum_{i=f(k,l-1)+1}^{f(k,l)} \sum_{\alpha=1}^{k} w_{i,\alpha}^* w_{i,\alpha} = 1.$$

We know that $M_k \otimes \mathcal{O}_n \subset \text{Im}(\varphi)$ because ψ_0 is surjective, so the sum $\sum_{\alpha=1}^{k} e_{1+\alpha,2} \otimes w_{i,\alpha}$ is in the image of φ for all $i \in \{f(k,l-1), \ldots, f(k,l)\}$. Hence

$$\begin{aligned}
&\sum_{i=f(k,l-1)+1}^{f(k,l)} \left(U_i^* \sum_{\alpha=1}^{k} e_{\alpha+1,2} \otimes w_{i,\alpha} \right) \\
&\qquad = \sum_{i=f(k,l-1)+1}^{f(k,l)} \left(\sum_{\alpha=1}^{k} e_{1,\alpha+1} \otimes w_{i,\alpha}^* \right) \left(\sum_{\alpha=1}^{k} e_{\alpha+1,2} \otimes w_{i,\alpha} \right) \\
&\qquad = \sum_{i=f(k,l-1)+1}^{f(k,l)} \sum_{\alpha=1}^{k} e_{1,2} \otimes w_{i,\alpha}^* w_{i,\alpha} \\
&\qquad = e_{1,2} \otimes \sum_{i=f(k,l-1)+1}^{f(k,l)} \sum_{\alpha=1}^{k} w_{i,\alpha}^* w_{i,\alpha} \\
&\qquad = e_{1,2} \otimes 1
\end{aligned}$$

is in the image of φ.

Now let $i \in \{3, \ldots, f(k,l)\}$. Then there exists $p \in \{1, \ldots, l-1\}$ such that $f(k,p)+1 \leq f(k,p+1)$ and unique integers q and r such that $i-1 = qk+r$ and $0 < r \leq k$. We know $q < f(k,l-1)$, as $q \geq f(k,l-1)$ implies

$$qk + r \geq f(k,l-1) \cdot k + r = f(k,l) - 1 + r \geq f(k,l) > i-1.$$

Thus

$$\varphi(e_{1,q+1} \otimes 1) = \sum_{\alpha=1}^{k} e_{\alpha+1,kq+\alpha+1} \otimes 1.$$

Surjectivity of ψ_0 implies there exists $a_i \in \mathcal{O}_n$ such that

$$\varphi(e_{1,1} \otimes a_i) = \psi_0(e_{1,1} \otimes a_i) = e_{2,r+1} \otimes 1.$$

Therefore

$$\begin{aligned}(e_{1,2}\otimes 1)\varphi(e_{1,1}\otimes a_i)\varphi(e_{1,q+1}\otimes 1) &= (e_{1,2}\otimes 1)(e_{2,r+1}\otimes 1)\left(\sum_{\alpha=1}^{k} e_{\alpha+1,qk+\alpha+1}\otimes 1\right)\\ &= (e_{1,2}\otimes 1)(e_{2,qk+r+1}\otimes 1)\\ &= (e_{1,2}\otimes 1)(e_{2,i}\otimes 1)\\ &= e_{1,i}\otimes 1\end{aligned}$$

is an element of $\mathrm{Im}(\varphi)$.

We now need only show that $e_{1,1}\otimes s_j \in \mathrm{Im}(\varphi)$ for all $j\in\{1,\dots,n\}$. Surjectivity of ψ_0 implies that there exists $b_j\in\mathcal{O}_n$ for each $j\in\{1,\dots,n\}$ such that

$$\varphi(e_{1,1}\otimes b_j)=\psi_0(e_{1,1}\otimes b_j)=e_{2,2}\otimes s_j.$$

Thus

$$(e_{1,2}\otimes 1)\varphi(e_{1,1}\otimes b_j)(e_{1,2}\otimes 1)^* = (e_{1,2}\otimes 1)(e_{2,2}\otimes s_j)(e_{2,1}\otimes 1)=e_{1,1}\otimes s_j$$

is in the image of φ for all $j\in\{1,\dots,n\}$. Hence φ is an automorphism. □

A slight modification of the above proof shows that Theorem 2.4 holds over $M_{f(l)}(F)\otimes L_n(F)$ as well.

2.3 Observing the order requirement

Under special circumstances, we can choose an isomorphism $\psi_0\colon M_1\otimes\mathcal{O}_n\to M_k\otimes\mathcal{O}_n$ (which can be regarded as choosing a set of V_j) and a set of U_i for $i\in\{f(k,l-1)+1,\dots,f(k,l)\}$ in the proof of Theorem 2.4 so that the resulting automorphism φ has order l. One such circumstance arises when $n=k^l=(k-1)f(k,l)$, and we will assume the following notation for the remainder of this section.

Notation 2.7. Let k and l be integers greater than 1 and set $n=k^l$.

We will henceforth assume the values of k and l are fixed. The section below defines 6 new integer functions, each depending on our choice of k and l in some way, and, if each of these dependencies were acknowledged, most mathematical expressions would be bloated by an excessive quantity of k's and l's that hinder rather than assist the reader. As a result, we redefine f to only depend on its second variable:

Notation 2.8. In the following section, we let f refer to a function on the positive integers given by

$$f(p) = \sum_{i=0}^{p-1} k^i.$$

This was formerly written as $f(k, p)$, and we omit the k for convenience of notation.

The reader will only be reminded of the above notation in major theorems and definitions; in lemmas and discussions, n, k, l, and f will be used without reference.

We now introduce the first of the 3 integer functions that allow us to construct this family of special automorphisms.

Definition 2.9. Let n, k, and l be as in Notation 2.7 and let f be as in Notation 2.8. Define three integer functions σ, λ, and μ by

$$\begin{aligned} \sigma(i) &= k\big(i - f(l-1) - 1\big), \\ \lambda(j) &= \left\lfloor \frac{j}{k^{l-1}} \right\rfloor, \\ \text{and} \qquad \mu(j) &= k(j \bmod k^{l-1}). \end{aligned}$$

We break up the main result into two parts: first, that the proposed formula defines a homomorphism which induces the automorphism $[1] \mapsto k \cdot [1]$ on K_0; and second, that this homomorphism is an automorphism of order l.

Theorem 2.10. Let n, k, and l be as in Notation 2.7 and let f be as in Notation 2.8. For all $i \in \{f(0), \ldots, f(l-1)\}$, set

$$U_i = \sum_{\beta=0}^{k-1} e_{\beta+f(1)+1,k(i-1)+\beta+f(1)+1} \otimes 1; \tag{2.1}$$

for all $i \in \{f(l-1)+1, \ldots, f(l)\}$, set

$$U_i = \sum_{\beta=0}^{k-1} e_{\beta+f(1)+1,1} \otimes s^*_{\sigma(i)+\beta+1}; \tag{2.2}$$

and for all $j \in \{0, \ldots, n-1\}$, set

$$V_{j+1} = \sum_{\beta=0}^{k-1} e_{\lambda(j)+f(1)+1,\beta+f(1)+1} \otimes s_{\mu(j)+\beta+1}. \tag{2.3}$$

Then there exists a unique homomorphism $\varphi\colon M_{f(l)} \otimes \mathcal{O}_n \to M_{f(l)} \otimes \mathcal{O}_n$ such that $\varphi(e_{1,i} \otimes 1) = U_i$ and $\varphi(e_{1,1} \otimes s_{j+1}) = V_{j+1}$ for all appropriate values of i and j. Moreover, $\varphi_*([1]) = k \cdot [1]$.

Although we could derive that φ is a homomorphism by using the construction from the proof of Theorem 2.4 and a particular choice of ψ_0 and U_i, it happens to be easier to directly apply Theorem 1.8. The connection between Theorems 2.4 and 2.10 will be discussed after proving that φ has order l.

Proof of Theorem 2.10. To show there is a unique homomorphism φ such that $\varphi(e_{1,i} \otimes 1) = U_i$ and $\varphi(e_{1,1} \otimes s_{j+1}) = V_{j+1}$ for all i and j, we must verify that

$$U_i U_j = 0 \qquad \text{if } i > 1 \tag{1}$$

$$U_i U_j^* = \begin{cases} U_1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{2}$$

$$\sum_{i=1}^{f(l)} U_i^* U_i = 1_{f(l)} \tag{3}$$

$$V_{i+1}^* V_{j+1} = \begin{cases} U_1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{4}$$

$$\sum_{j=0}^{n-1} V_{j+1} V_{j+1}^* = U_1 \tag{5}$$

for all appropriate values of i and j.

We begin by proving relation (1). Let $i, j \in \{1, \ldots, f(k,l)\}$ and suppose $i > 1$. If $i \leq$

$f(k, l-1)$, then

$$U_i = \sum_{\alpha=0}^{k-1} e_{\alpha+f(1)+1,k(i-1)+\alpha+f(1)+1} \otimes 1$$

and $k(i-1)+\alpha+f(1)+1 > \beta+f(1)+1$ for all $\alpha, \beta \in \{0, \ldots, k-1\}$. Alternatively, if $i > f(k, l-1)$, then

$$U_i = \sum_{\alpha=0}^{k-1} e_{\alpha+f(1)+1,1} \otimes s^*_{\sigma(i)+\alpha+1}$$

and $1 < \beta + f(1) + 1$ for all $\beta \in \{0, \ldots, k-1\}$. Thus

$$U_i \sum_{\beta=0}^{k-1} e_{\beta+f(1)+1,k(i-1)+\beta+f(1)+1} \otimes 1 = 0$$

and

$$U_i \sum_{\beta=0}^{k-1} e_{\beta+f(1)+1,1} \otimes s^*_{\sigma(i)+\beta+1} = 0$$

by definition of matrix unit multiplication. Thus $U_i U_j = 0$, proving relation (1).

We move on to proving relation (2). There are three cases to consider: one in which $1 \leq i, j \leq f(l-1)$, another where $1 \leq i \leq f(l-1) < j$, and one where $f(l-1) < i, j < f(l)$. The case when $1 \leq j < f(l-1) \leq i$ will be covered by the second case.

Case 1: $1 \leq i, j \leq f(l-1)$

In this case, the product of U_i and U_j^* is

$$U_i U_j^* = \left(\sum_{\alpha=0}^{k-1} e_{\alpha+f(1)+1,k(i-1)+\alpha+f(1)+1} \otimes 1 \right) \left(\sum_{\beta=0}^{k-1} e_{k(j-1)+\beta+f(1)+1,\beta+f(1)+1} \otimes 1 \right)$$

If $i \neq j$, then $k(i-1)+\alpha \neq k(j-1)+\beta$ for all $\alpha, \beta \in \{0, \ldots, k-1\}$ since $|i-j| \cdot k \geq k$. Thus $i \neq j$ implies $U_i U_j^* = 0$ by definition of matrix unit multiplication.

If instead $i = j$, then $k(i-1)+\alpha = k(j-1)+\beta$ for $\alpha, \beta \in \{0, \ldots, k-1\}$ precisely when $\alpha = \beta$, implying

$$U_i U_j^* = \sum_{\alpha=0}^{k-1} e_{\beta+f(1)+1,\beta+f(1)+1} \otimes 1 = U_1.$$

Thus relation (2) holds in case 1.

Case 2: $1 \leq i \leq f(l-1) < j$

In this case, the product of U_i and U_j^* is

$$U_iU_j^* = \left(\sum_{\alpha=0}^{k-1} e_{\alpha+f(1)+1,k(i-1)+\alpha+f(1)+1} \otimes 1\right)\left(\sum_{\beta=0}^{k-1} e_{1,\beta+f(1)+1} \otimes s_{\sigma(j)+\beta+1}\right).$$

Since $k(i-1)+\alpha \geq 0$, matrix multiplication implies that $U_iU_j^* = 0$. Taking the adjoint of both sides, we see that $U_jU_i^* = 0$, proving the analogous case $1 \leq j < k^{l-1} \leq i$.

Case 3: $f(l-1) < i, j \leq f(l)$

In this case, the product of U_i and U_j^* is

$$\begin{aligned} U_iU_j^* &= \left(\sum_{\alpha=0}^{k-1} e_{\alpha+f(1)+1,1} \otimes s^*_{\sigma(i)+\alpha+1}\right)\left(\sum_{\beta=0}^{k-1} e_{1,\beta+f(1)+1} \otimes s_{\sigma(j)+\beta+1}\right) \\ &= \sum_{\alpha,\beta=0}^{k-1} e_{\alpha+f(1)+1,\beta+f(1)+1} \otimes s^*_{\sigma(i)+\alpha+1}s_{\sigma(j)+\beta+1} \qquad (\star) \end{aligned}$$

If $i \neq j$, then $|\sigma(i)-\sigma(j)| \geq k$, so $\sigma(i)+\alpha \neq \sigma(j)+\beta$ for all $\alpha, \beta \in \{0, \ldots, k-1\}$. Thus $i \neq j$ implies $U_iU_j^* = 0$ by the first relation of $\mathcal{O}_n$.

On the other hand, if $i = j$, then $\sigma(i) = \sigma(j)$ and $\sigma(i)+\alpha = \sigma(j)+\beta$ precisely when $\alpha = \beta$, so

$$s^*_{\sigma(i)+\alpha+1}s_{\sigma(j)+\beta+1} = \delta_{\alpha,\beta}.$$

Continuing from $(\star)$, we then have

$$U_iU_j^* = \sum_{\alpha=0}^{k-1} e_{\alpha+f(1)+1,\alpha+f(1)+1} \otimes 1 = U_1.$$

Since the above three cases exhaust all possible choices of $i, j \in \{1, \ldots, f(l-1)\}$, this proves relation (2) for all such i and j.

There are a few preparatory calculations that we must perform to prove relation (3). As in relation (2), these calculations largely split between the case $i \leq f(l-1)$ and $i > f(l-1)$. Regardless of which of these inequalities hold, we prefer to express all $i \in \{1, \ldots, f(l)\}$ as a sum $1 + f(p) + j$ for some unique pair of integers $p \in \{0, \ldots, l-1\}$ and $j \in \{0, \ldots, k^p - 1\}$.

If $i \leq f(l-1)$ (or equivalently $p \leq l-2$), then

$$\begin{aligned} U_i^* U_i &= \left(\sum_{\alpha=0}^{k-1} e_{k(i-1)+\alpha+f(1)+1, \alpha+f(1)+1} \otimes 1 \right) \left(\sum_{\alpha=0}^{k-1} e_{\alpha+f(1)+1, k(i-1)+\alpha+f(1)+1} \otimes 1 \right) \\ &= \sum_{\alpha=0}^{k-1} e_{k(i-1)+\alpha+f(1)+1, k(i-1)+\alpha+f(1)+1} \otimes 1 \end{aligned} \tag{2.4}$$

by (2.1). Note that

$$k(i-1) + f(1) + 1 = k \cdot f(p) + kj + f(1) + 1 = 1 + f(p+1) + kj, \tag{2.5}$$

so we can uniquely express any integer between $1+f(p+1)$ and $f(p+2)$ as $k(i-1)+f(1)+\alpha+1$ for some $i \in \{1+f(p), \ldots, f(p+1)\}$ and some $\alpha \in \{0, \ldots, k-1\}$. Thus, if $p \leq l-2$, then

$$\begin{aligned} \sum_{i=1+f(p)}^{f(p+1)} U_i^* U_i &= \sum_{j=0}^{k^p-1} U_{1+f(p)+j}^* U_{1+f(p)+j} \\ &= \sum_{j=0}^{k^p-1} \sum_{\alpha=0}^{k-1} e_{1+f(p+1)+kj+\alpha, 1+f(p+1)+kj+\alpha} \otimes 1 \\ &= \sum_{i=1+f(p+1)}^{f(p+2)} e_{i,i} \otimes 1, \end{aligned}$$

where we combined (2.4) and (2.5) in the second step.

If instead $f(l-1) < i$ (or equivalently $p = l-1$), then

$$\begin{aligned} U_i^* U_i &= \left(\sum_{\alpha=0}^{k-1} e_{1,\beta+f(1)+1} \otimes s_{\sigma(i)+\beta+1} \right) \left(\sum_{\alpha=0}^{k-1} e_{\beta+f(1)+1,1} \otimes s_{\sigma(i)+\beta+1}^* \right) \\ &= \sum_{\alpha=0}^{k-1} e_{1,1} \otimes s_{\sigma(i)+\beta+1} s_{\sigma(i)+\beta+1}^* \end{aligned} \tag{2.6}$$

by (2.2). Note that

$$\sigma(i) + 1 = \sigma\big(1 + f(l-1) + j\big) + 1 = kj + 1. \tag{2.7}$$

Thus the division algorithm implies that every integer between 0 and $n-1$ is expressible as a sum $\sigma(i) + r + 1 = kj + r$ for unique integers $i \in \{1 + f(l-1), \ldots, f(l)\}$, $j \in \{0, \ldots, k^{l-1} - 1\}$ and

$r \in \{0, \ldots, k-1\}$. Therefore

$$\begin{aligned}
\sum_{i=1+f(l-1)}^{f(l)} U_i^* U_i &= \sum_{j=0}^{k^{l-1}-1} U_{1+f(l-1)+j}^* U_{1+f(l-1)+j} \\
&= \sum_{j=0}^{k^{l-1}-1} \sum_{\alpha=0}^{k-1} e_{1,1} \otimes s_{kj+\alpha+1} s_{kj+\alpha+1}^* \\
&= \sum_{j=0}^{n-1} e_{1,1} \otimes s_{j+1} s_{j+1}^* \\
&= e_{1,1} \otimes 1,
\end{aligned}$$

where we combined (2.6) and (2.7) in the second step and used the second relation of $\mathcal{O}_n$ in the last step. Thus

$$\begin{aligned}
\sum_{i=1}^{f(l-1)} U_i^* U_i &= \sum_{p=0}^{l-2} \sum_{j=0}^{k^p-1} U_{1+f(p)+j}^* U_{1+f(p)+j} + \sum_{j=0}^{k^{l-1}} U_{1+f(l-1)+j}^* U_{1+f(l-1)+j} \\
&= \sum_{p=0}^{l-2} \sum_{i=1+f(p+1)}^{f(p+2)} e_{i,i} \otimes 1 + e_{1,1} \otimes 1 \\
&= \sum_{p=1}^{l-1} \sum_{i=1+f(p)}^{f(p+1)} e_{i,i} \otimes 1 + e_{1,1} \otimes 1 \\
&= \sum_{p=0}^{l-1} \sum_{i=1+f(p)}^{f(p+1)} e_{i,i} \otimes 1 \\
&= \sum_{i=1}^{f(l)} e_{i,i} \otimes 1 = 1_{f(l)},
\end{aligned}$$

proving relation (3).

For relation (4), let $i, j \in \{0, \ldots, n-1\}$. As $0 \leq i, j < k^l = n$, it follows that both i and j can be expressed using no more than $l-1$ digits in base-k. Therefore i can be uniquely expressed as the sum $\lambda(i)k^{l-1} + \mu(i)/k$ and likewise for j, so $i = j$ if and only if $\lambda(i) = \lambda(j)$ and $\mu(i) = \mu(j)$.

Thus $i = j$ implies that

$$V_{i+1}^* V_{j+1} = \left(\sum_{\alpha=0}^{k-1} e_{\alpha+2,\lambda(i)+2} \otimes s_{\mu(j)+\alpha+1} \right) \left(\sum_{\beta=0}^{k-1} e_{\lambda(j)+2,\beta+2} \otimes s_{\mu(j)+\beta+1} \right)$$
$$= \sum_{\alpha,\beta=0}^{k-1} e_{\alpha+2,\beta+2} \otimes s_{\mu(i)+\alpha+1}^* s_{\mu(j)+\beta+1} \tag{2.8}$$

Since $\mu(i) = \mu(j)$, the indices $\mu(i) + \alpha + 1$ and $\mu(j) + \beta + 1$ coincide exactly when $\alpha = \beta$. Continuing from (2.8), we have

$$V_{i+1}^* V_{j+1} = \sum_{\beta=0}^{k-1} e_{\beta+2,\beta+2} \otimes 1$$

by the first relation of $\mathcal{O}_n$.

On the other hand, if $i \neq j$, then either $\lambda(i) \neq \lambda(j)$, in which case $V_{i+1}^* V_{j+1} = 0$ by matrix unit multiplication, or $\lambda(i) = \lambda(j)$, but $\mu(i)/k \neq \mu(j)/k$. Thus $|\mu(i) - \mu(j)| \geq k$, so

$$s_{\mu(i)+\alpha+1}^* s_{\mu(j)+\beta+1} = 0$$

for all $\alpha, \beta \in \{0, \ldots, k-1\}$ by relation 1 of $\mathcal{O}_n$. Continuing from (2.8) again, we have

$$V_{i+1} V_{j+1}^* = \sum_{\alpha,\beta=0}^{k-1} e_{\alpha+2,\beta+2} \otimes 0 = 0,$$

proving relation (4).

Finally, we check that relation (5) holds. Observe that each element $j \in \{0, \ldots, n-1\}$ can be uniquely expressed as a sum $qk^{l-1} + r$ for some $q \in \{0, \ldots, k-1\}$ and $r \in \{0, \ldots, k^{l-1}\}$, as well as a sum $kr + \beta$ for some $r \in \{0, \ldots, k-1\}$ and $\beta \in \{0, \ldots, k-1\}$. The first sum yields the identity

$$\sum_{j=0}^{n-1} V_{j+1} V_{j+1}^* = \sum_{q=0}^{k-1} \sum_{r=0}^{k^{l-1}-1} V_{qk^{l-1}+r+1} V_{qk^{l-1}+r+1}^*, \tag{2.9}$$

whereas the second sum yields the identity

$$\sum_{r=0}^{k^{l-1}-1}\sum_{\beta=0}^{k-1} s_{kr+\beta+1}s_{kr+\beta+1}^* = \sum_{j=0}^{n-1} s_{j+1}s_{j+1}^* = 1 \tag{2.10}$$

via the second relation of $\mathcal{O}_n$. Then, using (2.9) and (2.3) in the first step and (2.10) in the third,

$$\begin{aligned}
\sum_{j=0}^{n-1} V_{j+1}V_{j+1}^* &= \sum_{q=0}^{k-1}\sum_{r=0}^{k^{l-1}-1}\left(\sum_{\beta=0}^{k-1} e_{q+2,\beta+2}\otimes s_{kr+\beta+1}\right)\left(\sum_{\beta=0}^{k-1} e_{\beta+2,q+2}\otimes s_{kr+\beta+1}^*\right) \\
&= \sum_{q=0}^{k-1}\sum_{r=0}^{k^{l-1}-1}\sum_{\beta=0}^{k-1} e_{q+2,q+2}\otimes s_{kr+\beta+1}s_{kr+\beta+1}^* \\
&= \sum_{q=0}^{k-1} e_{q+2,q+2}\otimes 1 = U_1.
\end{aligned}$$

Thus relation (5) also holds, and there exists a unique homomorphism φ such that $\varphi(e_{1,i}\otimes 1) = U_i$ for all i and $\varphi(e_{1,1}\otimes s_{j+1}) = V_{j+1}$ for all j.

That $\varphi_*([1]) = k\cdot[1]$ follows from the fact that

$$[\varphi(e_{1,1}\otimes 1)] = [U_1] = \left[\sum_{\beta=0}^{k-1} e_{\beta+f(1)+1,\beta+f(1)+1}\otimes 1\right] = k\cdot[1].$$

□

The relations we chose to define $M_{f(l)}(\mathbb{C})$ imply that the above theorem holds over $M_{f(l)}(K)\otimes L_n(K)$ as well, where we replace each occurrence of s_i^* in the formula for φ with t_i.

Theorem 2.11. Let k, l, and n be as in Notation 2.7 and let f be as in Notation 2.8. The homomorphism $\varphi\colon M_{f(l)}\otimes\mathcal{O}_n\to M_{f(l)}\otimes\mathcal{O}_n$ defined in Theorem 2.10 is an automorphism and has order l.

Showing that φ^l is the identity map requires the definition of 3 more integer functions to help us describe where φ^m sends the generators of $M_{f(l)}\otimes\mathcal{O}_n$ for arbitrary $m\in\{1,\dots,l\}$. There are also several (perhaps confusing) choices of notation in the formula for φ which were made in the interest of calculating φ^m, which shall now be explained

Firstly, the choice to index the sums from 0 to $k-1$ rather than from 1 to k is one which better fits how the β indices are used in the calculation of φ^m. Each application of φ (up to a a point) will multiply the previous indices by k and add another index to the sum, so a total of m applications will yield a sum over m indices $\beta_0, \ldots, \beta_{m-1}$, usually of the form

$$\beta_0 k^{m-1} + \beta_1 k^{m-2} + \cdots + \beta_{m-2} k + \beta_{m-1}.$$

This provides the motivation for our fourth (family of) integer function(s):

Definition 2.12. For each $m \in \{0, \ldots, l-1\}$, define a function $F_m \colon \{0, \ldots, k-1\}^m \to \mathbb{Z}$ such that

$$F_m(\beta_0, \ldots, \beta_{m-1}) = \sum_{i=0}^{m-1} \beta_{m-i-1} k^i$$

We choose to denote these functions by F_m as they can be thought of a generalization of f. Note that, just as

$$f(m+1) = \sum_{i=1}^{m+1} k^i + 1 = k \cdot f(m) + f(1), \tag{2.11}$$

for all m, we also have a recursive relation

$$F_{m+1}(\beta_0, \ldots, \beta_m) = \sum_{i=1}^{m} \beta_{m-i} k^i + \beta_m = k \cdot F_m(\beta_0, \ldots, \beta_{m-1}) + \beta_m \tag{2.12}$$

that holds for all m and $\beta_0, \ldots, \beta_m \in \{0, \ldots, k-1\}$.

Although the β_i are the independent variables in the function F_m, it makes more sense for our purposes to consider $\beta_0, \ldots, \beta_{m-1}$ as coefficients of a polynomial, or as digits in the base-k representation of an integer. This last interpretation makes clear that F_m is a bijection between $\{0, \ldots, k-1\}^m$ and $\{0, \ldots, k^m - 1\}$, as each number in the latter range has a unique base-k representation $(\beta_0 \ldots \beta_{m-1})_k$. Indexing from 1 to k would shift the range up by 1 and result in mathematically equivalent expressions, but would also render the base-k analogy less useful and confuse the purpose of F_m.

Secondly, the expression $f(1) + 1 = k^0 + 1 = 2$ appears in the formula for φ in several places, with no clear reason as to why these are not replaced by 2's. This choice was also made to highlight the similarities between the formula for φ and the formula for φ^m, as these $f(1)$ terms will often be multiplied by k and then added to another $f(1)$ term, resulting in $f(2)$. Consequently, $\varphi^m(e_{1,i} \otimes 1)$ will have an $f(m)$ term in one of its indices regardless of i and m. Several formulas also involve subtracting 1 from an index, and the differentiation between $f(m)$ and 1 makes clear that the $f(m)$ term is of greater significance than the spare 1.

With these explanations out of the way, we turn our attention to the indices i and j themselves. We require specific data about i and j in order to make sense of $\sigma(i)$, $\lambda(j)$, and $\mu(j)$, as well as expressions derived from repeated applications of these functions. For example, if $i \in \{f(l-1)+1, \ldots, f(l)\}$, then $\varphi^2(e_{1,i} \otimes 1)$ involves the terms $\lambda(\sigma(i)+\beta_0)$ and $\mu(\sigma(i)+\beta_0)$. We thus settle on the following convention whenever we use terms i and j:

Notation 2.13. Let $j \in \{0, \ldots, n-1\}$. Then there exists unique integers $a_0, \ldots, a_{l-1} \in \{0, \ldots, k-1\}$ such that

$$j = \sum_{x=0}^{l-1} a_x k^x = (a_{l-1}a_{l-2}\ldots a_0)_k. \tag{2.13}$$

For all $i \in \{1, \ldots, f(l)\}$, there exists a unique integer $p \in \{0, \ldots, l-1\}$ such that $f(p)+1 \leq i \leq f(p+1)$. The difference between i and $f(p)+1$ is denoted by an integer $j \in \{0, \ldots, k^p - 1\}$ as in (2.13), so that

$$i = 1 + f(p) + j = 1 + f(p) + \sum_{x=0}^{p-1} a_x k^x, \tag{2.14}$$

where the last $l-p$ terms of j are omitted as $j < k^p$ implies that $a_x = 0$ for $x \geq p$.

Here j serves double duty as we want to consider the base-k representation of $i-1-f(p)$ for calculating $\varphi^m(e_{1,i} \otimes 1)$ as well as the base-k representation of an integer between 0 and $n-1$ for calculating $\varphi^m(e_{1,1} \otimes s_j)$. The author hopes that the reader will find this irksome convention more convenient than the usage of diacritics, as we are quickly running out of letters in the alphabet.

Letting j be as in Notation 2.13, one sees that

$$\lambda(j) = a_{l-1} \qquad \text{and} \qquad \mu(j) = \sum_{x=1}^{l-1} a_{x-1} k^x,$$

giving us a much better understanding of what λ and μ are doing: λ shifts j right $l-1$ times, whereas μ cuts off the top digit of j and then left-shifts the resulting number once. Thus the $(p-m)$-th digit of j is equal to $(\lambda \circ \mu^{m-1})(j)$.

The notation for i is a bit less intuitive, but can be understood by considering the formula for φ. If $i \leq f(l-1)$, then one uses (2.1) to calculate $\varphi(e_{1,i} \otimes 1)$. The matrix units in $\varphi(e_{1,i} \otimes 1)$ then contain the index

$$\begin{aligned}k(i-1)+f(1)+\beta_0+1 &= k\cdot f(p)+k\sum_{x=0}^{p-1}a_xk^x+f(1)+F_1(\beta_0)+1\\ &= 1+f(p+1)+\sum_{x=1}^{p}a_{x-1}k^x+F_1(\beta_0).\end{aligned}$$

If $p+1<l-1$, then the above expression would still be less than or equal to $f(l-1)$ because $F_1(\beta_0)<k$ and

$$\begin{aligned}1+f(p+1)+\sum_{x=1}^{p}a_{x-1}k^x &\leq 1+f(p+1)+k^{p+1}-k\\ &= f(p+2)-(k-1)\\ &\leq f(l-1)-(k-1).\end{aligned}$$

The equality in the second line uses the fact that $j\leq k^p-1$. Assuming the above inequalities hold, then $\varphi^3(e_{1,i}\otimes 1)$ would contain the index

$$\begin{aligned}&k\big(k(i-1)+F_1(\beta_0)+f(1)\big)+\beta_1+f(1)+1\\ &\qquad= k^2(i-1)+kF_1(\beta_0)+kf(1)+\beta_1+f(1)+1\\ &\qquad= k^2(i-1)+F_2(\beta_0,\beta_1)+f(2)+1\\ &\qquad= k^2f(p)+k^2\sum_{x=0}^{p}a_xk^x+F_2(\beta_0,\beta_1)+f(2)+1\\ &\qquad= 1+f(p+2)+\sum_{x=2}^{p}a_{x-2}k^x+F_2(\beta_0,\beta_1)\end{aligned}$$

and so on until $k^m(i-1)+F_m(\beta)+f(d)$ finally exceeds $f(l-1)$. As these indices all contain an $F_m(\beta_0,\dots,\beta_{m-1})$ term, we find it useful to shorten $(\beta_0,\dots,\beta_{m-1})$ to simply β where possible. Whenever another β_m index is added, we let (β,β_m) denote $(\beta_0,\dots,\beta_{m-1},\beta_m)$ to show that the m-tuple has changed into an $(m+1)$-tuple.

We must next ascertain when $k^m(i-1)+F_m(\beta)+f(m)\geq f(l-1)$, or, more precisely, find the minimal integer m satisfying $k^m(i-1)+F_m(\beta)+f(d)\geq f(l-1)$ for all β.

Lemma 2.14. If i, j, and p are as in Notation 2.13, then the difference $d=l-1-p$ is the minimal integer satisfying

$$f(l-1)+1 \leq k^m(i-1)+F_m(\beta)+f(m)+1 \leq f(l)$$

for all $\beta = (\beta_0, \ldots, \beta_{m-1}) \in \{0, \ldots, k-1\}^m$.

Proof. We first verify that d satisfies $f(l-1)+1 \leq k^d(i-1)+f(d)+1 \leq f(l)$. Observe that

$$k^d f(p) + f(d) = \sum_{x=d}^{p-d-1} k^x + \sum_{x=0}^{d-1} k^x = f(p-d) = f(l-1).$$

and

$$k^d\big(f(p+1)-1\big) + f(d) + 1 = f(p+d+1) - k^d + 1 \leq f(l).$$

The first inequality now follows from the assumption that $f(p)+1 \leq i$ and the second inequality from the assumption that $i \leq f(p+1)$. As $0 \leq F_d(\beta) < k^d \leq k^{l-1}$ for all β, it follows that

$$f(l-1)+1 \leq k^d(i-1)+F_d(\beta)+f(d)+1 \leq f(l).$$

Now suppose that $m < d$. As f is strictly increasing, we have

$$f(l-1) - \Big(k^m\big(f(p+1)-1\big) + f(m) + 1\Big) = f(p+d) - f(p+m+1) + k^m - 1 \geq k^m - 1.$$

The fact $f(l-1)$ differs from $k^m(i-1)+f(m)+1$ by at least k^m-1 means that the $F_m(\beta)$ term is curiously irrelevant in determining whether $k^m(i-1)+F_m(\beta)+f(m)+1$ is less than $f(l-1)$; the only relevant datum is the integer m. This proves the lemma. $\square$

We have now determined that $\varphi^d(e_{1,i} \otimes 1)$ contains a term of the form

$$\varphi(e_{1,k^d(i-1)+F_d(\beta)+f(d)+1} \otimes 1)$$

and that this term ought to be calculated using (2.2). The definition of σ implies that

$$\begin{aligned}
&\sigma\big(k^d(i-1)+F_d(\beta)+f(d)+1\big)+\beta_d+1 \\
&\qquad= k\big(1+f(l-1)+k^dj+F_d(\beta)-f(l-1)-1\big)+\beta_d+1 \\
&\qquad= \sum_{x=d+1}^{p+d} a_{x-d-1}k^x + k\cdot F_d(\beta)+\beta_{d+1}+1 \\
&\qquad= \sum_{x=d+1}^{l-1} a_{x-d-1}k^x + F_{d+1}(\beta,\beta_d)+1.
\end{aligned}$$

As this is an index of an s_{j+1} term, we now know $\varphi^{d+2}(e_{1,i}\otimes 1)$ contains terms of the form $\varphi(e_{1,1}\otimes s_{k^{d+1}j+F_{d+1}(\beta,\beta_d)+1})^*$. The next intuitive step is to calculate $\lambda(\cdot)$ and $\mu(\cdot)+\beta_{d+1}$ of the above expression minus 1. The definitions of λ and μ imply that

$$\begin{aligned}
\lambda\Big(\sigma\big(k^d(i-1)+F_d(\beta)+f(d)+1\big)+\beta_d\Big) &= \lambda\left(\sum_{x=d+1}^{l-1} a_{x-d-1}k^x + F_{d+1}(\beta,\beta_d)+1\right) \\
&= a_{l-1-d-1} = a_{p-1}
\end{aligned}$$

and

$$\begin{aligned}
&\mu\Big(\sigma\big(k^d(i-1)+F_d(\beta)+f(d)+1\big)+\beta_d\Big)+\beta_{d+1} \\
&\qquad= \mu\left(\sum_{x=d+1}^{l-1} a_{x-d-1}k^x + F_{d+1}(\beta,\beta_d)+1\right)+\beta_{d+1} \\
&\qquad= \sum_{x=d+2}^{l-1} a_{x-d-2}k^x + k\cdot F_{d+1}(\beta,\beta_d)+\beta_{d+1} \\
&\qquad= \sum_{x=d+2}^{l-1} a_{x-d-2}k^x + F_{d+2}(\beta,\beta_{d+1}).
\end{aligned}$$

At long last we are able to see a sliver of the picture with some clarity. Repeated applications of $\mu(\cdot)+\beta_{m+1}$ will eventually yield a term $e_{1,1}\otimes s_{F_l(\beta)+1}$ that ranges from 1 to n and will likely (the author is cheating with hindsight) cancel out with $e_{1,1}\otimes s^*_{F_l(\beta)+1}$ terms to form a single $e_{1,1}\otimes 1$ term by one of the defining relations of $\mathcal{O}_n$. Applying λ to indices like $k^dj+F_{d+1}(\beta,\beta_d)+1$ or $\mu(k^dj+F_{d+1}(\beta,\beta_d)+1)$ peels off the top digit of j, which is then multiplied by k as dictated by

(2.1). This process eventually yields a "reconstructed" copy of $e_{1,j+f(p)+1} \otimes 1 = e_{1,i} \otimes 1$.

These claims might seem speculative at the moment, but will soon be substantiated by rigorous proof. The point of all these calculations was to motivate the definitions of the last 2 integer functions.

Definition 2.15. Let i, j, and p be as expressed in Notation 2.13. Define a function $A_j \colon \{1, \ldots, l\} \to \mathbb{Z}$ by

$$A_j(m) = \sum_{x=0}^{m-1} a_{l-m+x} k^x = a_{l-1} k^{m-1} + \cdots + a_{l-m+1} k + a_{l-m}.$$

Furthermore, for each $m \in \{1, \ldots, p\}$, define a function $S_{j,m} \colon \{0, 1, \ldots, k-1\}^m \to \mathbb{Z}$ such that

$$\begin{aligned} S_{j,m}(\beta_0, \ldots, \beta_{m-1}) &= \sum_{j=m}^{l-1} a_{j-m} k^j + \sum_{j=0}^{m-1} \beta_{m-j-1} k^j \\ &= a_{l-m-1} k^{p-1} + \cdots + a_0 k^m + F_m(\beta_0, \ldots, \beta_{m-1}). \end{aligned}$$

As with F_m, we will often shorten $S_{j,m}(\beta_0, \ldots, \beta_{m-1})$ to $S_{j,m}(\beta)$ to save space.

One should see some similarities between the definitions of A_j and $S_{j,m}$ and the calculations involving λ and μ above. The function A_j models the reconstruction of j via repeated applications of λ and μ to $k^d j + F_{d+1}(\beta) + 1$, and $S_{j,m}$ models how the $e_{1,1} \otimes s_{k^d j + F_d(\beta)+1}$ terms are eventually replaced by $e_{1,1} \otimes s_{F_l(\beta)}$ terms through repeated applications of $\mu(\cdot) + \beta_m$. Wordy explanations are unlikely to satisfy any unresolved confusion at this point, so we shall henceforth stick to calculations and terse descriptions.

Both A_j and $S_{j,m}$ possess recursive properties that appear several times in the proof that φ has order l, so we take one last detour before we break up the proof of Theorem 2.11 into two more manageable lemmas.

Lemma 2.16. Let i and p be as expressed in Notation 2.13, let $m \in \{1, \ldots, p-1\}$, and let A_j and $S_{j,m}$ be as in Definition 2.15. Then

$$A_j(m+1) = k \cdot A_j(m) + a_{p-m-1} \tag{2.15}$$

and

$$S_{j,m+1}(\beta, \beta_m) = \mu(S_{j,m}(\beta)) + \beta_m, \tag{2.16}$$

for all $\beta = (\beta_0, \ldots, \beta_{m-1}) \in \{0, 1, \ldots, k-1\}^m$ and $\beta_m \in \{0, 1, \ldots, k-1\}$.

Proof. Let $m \in \{1, \ldots, l-1\}$. Both properties follow directly from Definition 2.15, as shall soon be made apparent. For the first equation, we calculate:

$$k \cdot A_j(m) + a_{p-m-1} = \sum_{j=0}^{m-1} a_{p-m+j} k^{j+1} + a_{p-m-1} = \sum_{j=1}^{m} a_{p-(m+1)+j} k^{p} + a_{p-(m+1)} = A_j(m+1).$$

And for the second equation we calculate:

$$\begin{aligned}
\mu\big(S_{j,m}(\beta)\big) + \beta_m &= \sum_{j=m}^{p-2} a_{j-m} k^{j+1} + k \cdot K_m(\beta) + \beta_m + 1 \\
&= \sum_{j=m+1}^{p-1} a_{j-(m+1)} k^{j} + K_{m+1}(\beta, \beta_m) + 1 \\
&= S_{j,m+1}(\beta, \beta_m).
\end{aligned}$$

This completes the proof. $\square$

We now have the necessary tools to begin describing where φ^m sends $e_{1,i} \otimes 1$ for various values of m.

Lemma 2.17. Let i, j, and p be as in Notation 2.13, let d be as in Lemma 2.14, and let φ be as in Theorem 2.10. Then for $m \in \{1, \ldots, d\}$, we have

$$\varphi^m(e_{1,i} \otimes 1) = \sum_{\beta_0, \ldots, \beta_{m-1}=0}^{k-1} e_{F_m(\beta)+f(m)+1,\, k^m(i-1)+F_m(\beta)+f(m)+1} \otimes 1,$$

and for $m \in \{d+1, \ldots, l-1\}$

$$\varphi^m(e_{1,i} \otimes 1) = \sum_{\beta_0, \ldots, \beta_{m-1}=0}^{k-1} e_{F_m(\beta)+f(m)+1,\, A_j(m)+f(m-d-1)+1} \otimes s^*_{S_{j,m}(\beta)+1}$$

Proof. We prove both assertions by induction. Suppose for the first assertion that $d > 0$ or, equivalently, that $i \leq f(l-1)$. Then the base case of the first assertion follows directly from equation (2.1) in the definition of φ in Theorem 2.10.

Now suppose that, for some $m \in \{1, \ldots, d-1\}$, we have

$$\varphi^m(e_{1,i} \otimes 1) = \sum_{\beta_0,\ldots,\beta_{m-1}=0}^{k-1} e_{F_m(\beta)+f(m)+1,k^m(i-1)+F_m(\beta)+f(m)+1} \otimes 1.$$

We now set out to calculate

$$\begin{aligned}
&\varphi(e_{F_m(\beta)+f(m)+1,k^m(i-1)+F_m(\beta)+f(m)+1} \otimes 1) \\
&\quad = \varphi(e_{1,F_m(\beta)+f(m)+1} \otimes 1)^* \varphi(e_{1,k^m(i-1)+F_m(\beta)+f(m)+1} \otimes 1)
\end{aligned}$$

for a fixed, but arbitrary m-tuple β. The proof of Lemma 2.14 shows that both $F_m(\beta) + f(m) + 1$ and $k^m(i-1) + F_m(\beta) + f(m) + 1$ are less than $f(l-1)$. Equation (2.1) implies that

$$\begin{aligned}
\varphi(e_{1,F_m(\beta)+f(m)+1} \otimes 1)^* &= \sum_{\beta_m=0}^{k-1} e_{k(F_m(\beta)+f(m))+\beta_m+f(1)+1,\beta_m+f(1)+1} \otimes 1 \\
&= \sum_{\beta_m=0}^{k-1} e_{F_{m+1}(\beta,\beta_m)+f(m+1)+1,\beta_m+f(1)+1} \otimes 1. \qquad (2.17)
\end{aligned}$$

The simplification of the left index follows from equations (2.11) and (2.12). Likewise, we have

$$\begin{aligned}
\varphi(e_{1,k^m(i-1)+F_m(\beta)+f(m)+1} \otimes 1) &= \sum_{\beta_m=0}^{k-1} e_{\beta_m+f(1)+1,k(k^m(i-1)+F_m(\beta)+f(m))+\beta_m+f(1)+1} \otimes 1 \\
&= \sum_{\beta_m=0}^{k-1} e_{\beta_m+f(1)+1,k^{m+1}(i-1)+F_{m+1}(\beta,\beta_m)+f(m+1)+1} \otimes 1
\end{aligned}$$

(2.18)

from (2.1), (2.11), and (2.12). The product of (2.17) and (2.18) is

$$\begin{aligned}
&\varphi(e_{F_m(\beta)+f(m)+1,k^m(i-1)+F_m(\beta)+f(m)+1} \otimes 1) \\
&\quad = \varphi(e_{1,k^m(i-1)+F_m(\beta)+f(m)+1} \otimes 1)^* \varphi(e_{1,k^m(i-1)+F_m(\beta)+f(m)+1} \otimes 1) \\
&\quad = \left(\sum_{\beta_m=0}^{k-1} e_{F_{m+1}(\beta,\beta_m)+f(m+1)+1,\beta_m+f(1)+1} \otimes 1 \right) \\
&\qquad \left(\sum_{\beta_m=0}^{k-1} e_{\beta_m+f(1)+1,k^{m+1}(i-1)+F_{m+1}(\beta,\beta_m)+f(m+1)+1} \otimes 1 \right) \\
&\quad = \sum_{\beta_m=0}^{k-1} e_{F_{m+1}(\beta,\beta_m)+f(m+1)+1,k^{m+1}(i-1)+F_{m+1}(\beta,\beta_m)+f(m+1)+1} \otimes 1.
\end{aligned}$$

Hence

$$\begin{aligned}
\varphi^{m+1}(e_{1,i} \otimes 1) &= \sum_{\beta_0,\ldots,\beta_{m-1}=0}^{k-1} \varphi(e_{F_m(\beta)+f(m)+1,k^m(i-1)+F_m(\beta)+f(m)+1} \otimes 1) \\
&= \sum_{\beta_1,\ldots,\beta_m=0}^{k-1} e_{F_{m+1}(\beta,\beta_m)+f(m+1)+1,k^{m+1}(i-1)+F_{m+1}(\beta,\beta_m)+f(m+1)+1} \otimes 1,
\end{aligned}$$

proving the first part of the lemma.

For the second part, we must consider two base cases, one where $0 < d < l-1$ and the first assertion holds, and another where $d = 0$ and the first assertion is vacuous. The $d = 0$ base case follows directly from equation (2.2) and the definitions of f, F_m, A_j, and $S_{j,m}$, from which we derive the identities

$$\begin{aligned}
f(1) &= 1, \\
F_1(\beta_0) &= \beta_0, \\
A_j(1) &= 0, \\
S_{j,1}(\beta_0) &= k\sum_{x=0}^{l-2} a_x k^x + F_1(\beta_0) = \sum_{x=1}^{l-1} a_x k^x + \beta_0.
\end{aligned}$$

Thus we need only consider the $0 < d < l-1$ base case, where we have

$$\varphi^d(e_{1,i} \otimes 1) = \sum_{\beta_0,\ldots,\beta_{d-1}=0}^{k-1} e_{F_d(\beta)+f(d)+1,k^d(i-1)+F_d(\beta)+f(d)+1} \otimes 1.$$

We set out to calculate

$$\begin{aligned}&\varphi(e_{F_d(\beta)+f(d)+1,k^d(i-1)+F_d(\beta)+f(d)+1} \otimes 1)\\&\quad = \varphi(e_{1,F_d(\beta)+f(d)+1} \otimes 1)^* \varphi(e_{1,k^d(i-1)+F_d(\beta)+f(d)+1} \otimes 1)\end{aligned}$$

for a fixed, but arbitrary m-tuple β. As $F_d(\beta) < k^d$ for all β, we have

$$F_d(\beta) + f(d) + 1 \leq k^d - 1 + f(d) + 1 = f(d+1) \leq f(l-1).$$

Thus we reuse the result our previous calculation in equation (2.17) for $\varphi(e_{1,F_d(\beta)+f(d)+1} \otimes 1)^*$. Lemma 2.14 implies that $f(l-1)+1 \leq k^d(i-1)+f(d)+1$, so equation (2.2) provides the formula for calculating $\varphi(e_{1,k^d(i-1)+F_d(\beta)+f(d)+1} \otimes 1)$. Recall from our earlier calculations that

$$\begin{aligned}\sigma\big(k^d(i-1) + F_d(\beta) + f(d) + 1\big) + \beta_d &= \sum_{x=d+1}^{l-1} a_{x-d-1} k^x + F_{d+1}(\beta, \beta_d)\\&= S_{j,d+1}(\beta, \beta_d)\end{aligned}$$

for all $\beta_d \in \{0, \ldots, k-1\}$. Therefore equation (2.2) implies that

$$\begin{aligned}\varphi(e_{1,k^d(i-1)+F_d(\beta)+f(d)+1} \otimes 1) &= \sum_{\beta_{d+1}=0}^{k-1} e_{\beta_{d+1}+f(1)+1,1} \otimes s_{\sigma(k^d(i-1)+F_d(\beta)+f(d)+1)+\beta_m+1}\\&= \sum_{\beta_{d+1}=0}^{k-1} e_{\beta_{d+1}+f(1)+1,1} \otimes s_{S_{j,d+1}(\beta,\beta_d)+1}. \qquad (2.19)\end{aligned}$$

Multiplying (2.17) and (2.19) yields

$$\begin{aligned}
&\varphi(e_{F_d(\beta)+f(d)+1,k^d(i-1)+F_d(\beta)+f(d)+1} \otimes 1) \\
&\quad = \varphi(e_{1,F_d(\beta)+f(d)+1} \otimes 1)^* \varphi(e_{1,k^d(i-1)+F_d(\beta)+f(d)+1} \otimes 1) \\
&\quad = \left(\sum_{\beta_d=0}^{k-1} e_{F_{d+1}(\beta,\beta_d)+f(d+1)+1,\beta_d+f(1)+1} \otimes 1 \right) \\
&\qquad \left(\sum_{\beta_{d+1}=0}^{k-1} e_{\beta_{d+1}+f(1)+1,1} \otimes s_{S_{j,d+1}(\beta,\beta_d)+1} \right) \\
&\quad = \sum_{\beta_d=0}^{k-1} e_{F_{d+1}(\beta,\beta_d)+f(d+1)+1,1} \otimes s_{S_{j,d+1}(\beta,\beta_d)+1}.
\end{aligned}$$

We know $f(0) = 0$; that $A_j(1) = a_{l-1} = 0$ as well follows from $j < k^{l-1}$, which implies the $(l-1)$-th digit of j in its base-k representation is 0. The above calculation thus proves the base case of the second assertion.

Now suppose that, for some $m \in \{d+1, \dots, l-2\}$, we have

$$\varphi^m(e_{1,i} \otimes 1) = \sum_{\beta_0,\dots,\beta_{m-1}=0}^{k-1} e_{F_m(\beta)+f(m)+1,A_j(m)+f(m-d-1)+1} \otimes s^*_{S_{j,m}(\beta)+1}.$$

We aim to calculate

$$\begin{aligned}
&\varphi(e_{F_m(\beta)+f(m)+1,A_j(m)+f(m-d-1)+1} \otimes s^*_{S_{j,m}(\beta)+1}) \\
&\quad = \varphi(e_{1,F_m(\beta)+f(m)+1} \otimes 1)^* \varphi(e_{1,1} \otimes s_{S_{j,m}(\beta)+1})^* \varphi(e_{1,A_j(m)+f(m-d-1)+1} \otimes 1)
\end{aligned}$$

for fixed β. As $m < l-1$ and $F_m(\beta) < k^m$, we have

$$F_m(\beta) + f(m) + 1 \leq k^m - 1 + f(m) + 1 = f(m+1) \leq f(l-1),$$

so we ought to calculate $\varphi(e_{1,F_m(\beta)+f(m)+1} \otimes 1)^*$ using equation (2.1). Thus we may reuse (2.17) again for $\varphi(e_{1,F_m(\beta)+f(m)+1} \otimes 1)^*$.

For $A_j(m) + f(m-d-1)$, recall that $j < k^p = k^{l-d-1}$ implies that $a_x = 0$ for $x \in$

$\{l-d-1,\ldots,l-1\}$. Hence $A_j(m) < k^{l-d-1}$, so

$$A_j(m) + f(m-d-1) < k^{m-d-1} + f(m-d-1) = f(m-d) \leq f(p-1) < f(l-1).$$

Equations (2.1), (2.15), and (2.11) give us

$$\begin{aligned}\varphi(e_{1,A_j(m)+f(m-d-1)+1} \otimes 1) &= \sum_{\beta_m=0}^{k-1} e_{\beta_m+f(1)+1,k(A_j(m)+f(m-d-1))+\beta_m+f(1)+1} \otimes 1 \\ &= \sum_{\beta_m=0}^{k-1} e_{\beta_m+f(1)+1,k\cdot A_j(m)+f(m-d)+\beta_m+1} \otimes 1. \end{aligned} \tag{2.20}$$

Before using equation (2.3) to calculate $\varphi(e_{1,1} \otimes s_{S_{j,m}(\beta)+1})^*$, recall that $\lambda(S_{j,m}(\beta)) = a_{l-m-1}$ and $\mu(S_{j,m}(\beta)) + \beta_m = S_{j,m+1}(\beta, \beta_m)$. The first of these follows from the definition of λ and $S_{j,m}$ and the second comes from (2.16) in Lemma 2.16. Therefore

$$\begin{aligned}\varphi(e_{1,1} \otimes s_{S_{j,m}(\beta)+1})^* &= \sum_{\beta_m=0}^{k-1} e_{\beta_m+f(1)+1,\lambda(S_{j,m}(\beta)+f(1)+1} \otimes s^*_{\mu(S_{j,m}(\beta)+\beta_m+1} \\ &= \sum_{\beta_m=0}^{k-1} e_{\beta_m+f(1)+1,a_{l-m-1}+f(1)+1} \otimes s^*_{S_{j,m+1}(\beta,\beta_m)+1}. \end{aligned} \tag{2.21}$$

Multiplying (2.17) and (2.21) together yields

$$\begin{aligned}&\varphi(e_{F_m(\beta)+f(m)+1,1} \otimes s^*_{S_{j,m}(\beta)+1}) \\ &\quad = \varphi(e_{1,F_m(\beta)+f(m)+1} \otimes 1)^* \varphi(e_{1,1} \otimes s_{S_{j,m}(\beta)+1})^* \\ &\quad = \left(\sum_{\beta_m=0}^{k-1} e_{F_{m+1}(\beta,\beta_m)+f(m+1)+1,\beta_m+f(1)+1} \otimes 1 \right) \\ &\qquad \left(\sum_{\beta_m=0}^{k-1} e_{\beta_m+f(1)+1,a_{l-m-1}+f(1)+1} \otimes s^*_{S_{j,m+1}(\beta,\beta_m)+1} \right) \\ &\quad = \sum_{\beta_m=0}^{k-1} e_{F_{m+1}(\beta,\beta_m)+f(m+1)+1,a_{l-m-1}+f(1)+1} \otimes s^*_{S_{j,m+1}(\beta,\beta_m)+1}. \end{aligned}$$

Right-multiplying the above product with (2.20) results in the sum

$$\begin{aligned}
&\varphi(e_{F_m(\beta)+f(m)+1,A_j(m-d-1)} \otimes s^*_{S_{j,m}(\beta)+1}) \\
&\quad = \varphi(e_{F_m(\beta)+f(m)+1,1} \otimes s^*_{S_{j,m}(\beta)+1})\varphi(e_{1,A_j(m-d-1)+1} \otimes 1) \\
&\quad = \left(\sum_{\beta_m=0}^{k-1} e_{F_{m+1}(\beta,\beta_m)+f(m+1)+1,a_{l-m-1}+f(1)+1} \otimes s^*_{S_{j,m+1}(\beta,\beta_m)+1} \right) \\
&\qquad \left(\sum_{\beta_m=0}^{k-1} e_{\beta_m+f(1)+1,k\cdot A_j(m)+f(m-d)+\beta_m+1} \otimes 1 \right) \\
&\quad = \sum_{\beta_m=0}^{k-1} e_{F_{m+1}(\beta,\beta_m)+f(m+1)+1,k\cdot A_j(m)+f(m-d)+a_{l-m-1}+1} \otimes s^*_{S_{j,m+1}(\beta,\beta_m)+1} \\
&\quad = \sum_{\beta_m=0}^{k-1} e_{F_{m+1}(\beta,\beta_m)+f(m+1)+1,A_j(m)+f(m-d)+1} \otimes s^*_{S_{j,m+1}(\beta,\beta_m)+1},
\end{aligned}$$

where the last step follows from (2.15) in Lemma 2.15. Hence

$$\begin{aligned}
\varphi^{m+1}(e_{1,i} \otimes 1) &= \sum_{\beta_0,\ldots,\beta_{m-1}=0}^{k-1} \varphi(e_{F_m(\beta)+f(m)+1,A_j(m)} \otimes s^*_{S_{j,m}(\beta)+1}) \\
&= \sum_{\beta_0,\ldots,\beta_m=0}^{k-1} e_{F_{m+1}(\beta,\beta_m)+f(m+1)+1,A_j(m)+f(m-d)+1} \otimes s^*_{S_{j,m+1}(\beta,\beta_m)+1},
\end{aligned}$$

proving the second assertion and the lemma. □

Lemma 2.17 allow us to calculate $\varphi^l(e_{1,i} \otimes 1)$, as we now have explicit formulas for $\varphi^{l-1}(e_{1,i} \otimes 1)$ for all i. Before doing so, we prove the corresponding version of Lemma 2.17 for $e_{1,1} \otimes s_{j+1}$.

Lemma 2.18. Let j be as in Notation 2.13 and let φ be as in Theorem 2.10. Then

$$\varphi^m(e_{1,1} \otimes s_{j+1}) = \sum_{\beta_0,\ldots,\beta_{m-1}=0}^{k-1} e_{A_j(m)+f(m)+1,F_m(\beta)+f(m)+1} \otimes s_{S_{j,m}(\beta)+1}$$

for all $m \in \{1, \ldots, l-1\}$.

Proof. We proceed by induction. The base case follows directly from equation (2.3) and the fact that $A_j(1) = a_{l-1} = \lambda(j)$, $F_1(\beta_0) = \beta_0$, and $\mu(j) + \beta_0 = S_{j,1}(\beta_0)$.

Suppose that, for some $m \in \{1, \ldots, l-2\}$, we have

$$\varphi^m(e_{1,1} \otimes s_{j+1}) = \sum_{\beta_0,\ldots,\beta_{m-1}=0}^{k-1} e_{A_j(m)+f(m)+1,F_m(\beta)+f(m)+1} \otimes s_{S_{j,m}(\beta)}.$$

We set out to calculate

$$\begin{aligned}
&\varphi(e_{A_j(m)+f(m)+1,F_m(\beta)+f(m)+1} \otimes s_{S_{j,m}(\beta)+1}) \\
&\quad = \varphi(e_{1,A_j(m)+f(m)+1} \otimes 1)^* \varphi(e_{1,1} \otimes s_{S_{j,m}(\beta)+1}) \varphi(e_{1,F_m(\beta)+f(m)+1} \otimes 1)
\end{aligned}$$

for fixed, but arbitrary β. Both $A_j(m)$ and $F_m(\beta)$ are bounded from above by $k^m - 1$, so $A_j(m) + f(m)$ and $F_m(\beta) + f(m)1$ are less than $f(l-1)$. Thus both $\varphi(e_{1,A_j(m)+f(m)+1} \otimes 1)^*$ and $\varphi(e_{1,F_m(\beta)+f(m)+1} \otimes 1)$ are calculated using (2.1). In fact, we have already calculated the adjoint of $\varphi(e_{1,F_m(\beta)+f(m)+1} \otimes 1)$ in (2.17), so we can reuse this equation once again by simply reversing the indices. Thus

$$\begin{aligned}
\varphi(e_{1,A_j(m)+f(m)+1} \otimes 1)^* &= \sum_{\beta_m=0}^{k-1} e_{k(A_j(m)+f(m))+\beta_m+f(1)+1,\beta_m+f(1)+1} \otimes 1 \\
&= \sum_{\beta_m=0}^{k-1} e_{k \cdot A_j(m)+\beta_m+f(m+1)+1,\beta_m+f(1)+1} \otimes 1 \qquad (2.22)
\end{aligned}$$

by (2.1) and (2.11), and

$$\varphi(e_{1,F_m(\beta)+f(m)+1} \otimes 1) = \sum_{\beta_m=0}^{k-1} e_{\beta_m+f(1)+1,F_{m+1}(\beta,\beta_m)+f(m+1)+1} \otimes 1. \qquad (2.23)$$

by (2.17). For the middle term we use (2.15), (2.16) and (2.3) to calculate:

$$\begin{aligned}
\varphi(e_{1,1} \otimes s_{S_{j,m}(\beta)+1}) &= \sum_{\beta_m=0}^{k-1} e_{\lambda(S_{j,m})+f(1)+1,\beta_m+f(1)+1} \otimes s_{\mu(S_{j,m}(\beta)+\beta_m+1} \\
&= \sum_{\beta_m=0}^{k-1} e_{a_{l-m-1}+f(1)+1,\beta_m+f(1)+1} \otimes s_{S_{j,m+1}(\beta,\beta_m)+1}. \qquad (2.24)
\end{aligned}$$

Multiplying (2.22) and (2.24) together yields

$$\begin{aligned}
&\varphi(e_{A_j(m)+f(m)+1,1} \otimes s_{S_{j,m}(\beta)+1}) \\
&\quad = \varphi(e_{1,A_j(m)+f(m)+1} \otimes 1)^* \varphi(e_{1,1} \otimes s_{S_{j,m}(\beta)+1}) \\
&\quad = \left(\sum_{\beta_m=0}^{k-1} e_{k\cdot A_j(m)+\beta_m+f(m+1)+1,\beta_m+f(1)+1} \otimes 1 \right) \\
&\qquad \left(\sum_{\beta_m=0}^{k-1} e_{a_{l-m-1}+f(1)+1,\beta_m+f(1)+1} \otimes s_{S_{j,m+1}(\beta,\beta_m)+1} \right) \\
&\quad = \sum_{\beta_m=0}^{k-1} e_{k\cdot A_j(m)+a_{l-m-1}+f(m+1)+1,\beta_m+f(1)+1} \otimes s_{S_{j,m+1}(\beta,\beta_m)+1} \\
&\quad = \sum_{\beta_m=0}^{k-1} e_{A_j(m+1)+f(m+1)+1,\beta_m+f(1)+1} \otimes s_{S_{j,m+1}(\beta,\beta_m)+1},
\end{aligned}$$

where the last step uses (2.15) from Lemma 2.16. Right-multiplying the above product by (2.18) results in the sum

$$\begin{aligned}
&\varphi(e_{A_j(m)+f(m)+1,F_m(\beta)+f(m)+1} \otimes s_{S_{j,m}(\beta)+1}) \\
&\quad = \varphi(e_{A_j(m)+f(m)+1,1} \otimes s_{S_{j,m}(\beta)+1}) \varphi(e_{1,F_m(\beta)+f(m)+1} \otimes 1) \\
&\quad = \left(\sum_{\beta_m=0}^{k-1} e_{A_j(m+1)+f(m+1)+1,\beta_m+f(1)+1} \otimes s_{S_{j,m+1}(\beta,\beta_m)+1} \right) \\
&\qquad \left(\sum_{\beta_m=0}^{k-1} e_{\beta_m+f(1)+1,F_{m+1}(\beta,\beta_m)+f(m+1)+1} \otimes 1 \right) \\
&\quad = \sum_{\beta_m=0}^{k-1} e_{A_j(m+1)+f(m+1)+1,F_{m+1}(\beta,\beta_m)+f(m+1)+1} \otimes s_{S_{j,m+1}(\beta,\beta_m)+1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\varphi^{m+1}(e_{1,1} \otimes s_{j+1}) &= \sum_{\beta_0,\ldots,\beta_{m-1}=0}^{k-1} \varphi(e_{A_j(m)+f(m)+1,F_m(\beta)+f(m)+1} \otimes s_{S_{j,m}(\beta)+1}) \\
&= \sum_{\beta_0,\ldots,\beta_m=0}^{k-1} e_{A_j(m+1)+f(m+1)+1,F_{m+1}(\beta,\beta_m)+f(m+1)+1} \otimes s_{S_{j,m+1}(\beta,\beta_m)+1},
\end{aligned}$$

proving the lemma. □

All that remains is to prove Theorem 2.11 itself.

Proof of Theorem 2.11. Let i, p, and j be as in Notation 2.13. We claim that $\varphi^l(e_{1,i}\otimes 1) = e_{1,i}\otimes 1$. We first take care of the special case where $i = 1$, $p = 0$, and $j = 0$. According to Lemma 2.17, we have

$$\varphi^{l-1}(e_{1,1}\otimes 1) = \sum_{\beta_0,\ldots,\beta_{l-2}=0}^{k-1} e_{F_{l-1}(\beta)+f(l-1)+1,F_{l-1}(\beta)+f(l-1)+1}\otimes 1.$$

Temporarily fix β. Clearly $F_{l-1}(\beta) + f(l-1) + 1 \geq f(l-1) + 1$, so we use equation (2.2) to calculate $\varphi(e_{1,F_{l-1}(\beta)+f(l-1)+1}\otimes 1)$ and its adjoint. Note that

$$\sigma\big(F_{l-1}(\beta) + f(l-1) + 1\big) + \beta_{l-1} = kF_{l-1}(\beta) + \beta_{l-1} = F_l(\beta, \beta_{l-1}).$$

Thus

$$\begin{aligned}
\varphi(e_{1,F_{l-1}(\beta)+f(l-1)+1}\otimes 1) &= \sum_{\beta_{l-1}=0}^{k-1} e_{\beta_{l-1}+f(1),1}\otimes s^*_{\sigma(F_{l-1}(\beta)+f(l-1)+1)+\beta_{l-1}+1}\\
&= \sum_{\beta_{l-1}=0}^{k-1} e_{\beta_{l-1}+f(1)+1,1}\otimes s^*_{F_l(\beta,\beta_{l-1})+1}. \qquad (2.25)
\end{aligned}$$

Left-multiplying (2.25) by its adjoint yields the product

$$\begin{aligned}
&\varphi(e_{F_{l-1}(\beta)+f(l-1)+1,F_{l-1}(\beta)+f(l-1)+1}\otimes 1)\\
&\quad= \varphi(e_{1,F_{l-1}(\beta)+f(l-1)+1}\otimes 1)^*\varphi(e_{1,F_{l-1}(\beta)+f(l-1)+1}\otimes 1)\\
&\quad= \left(\sum_{\beta_{l-1}=0}^{k-1} e_{1,\beta_{l-1}+f(1)}\otimes s_{F_l(\beta,\beta_{l-1})+1}\right)\\
&\qquad\left(\sum_{\beta_{l-1}=0}^{k-1} e_{\beta_{l-1}+f(1),1}\otimes s^*_{F_l(\beta,\beta_{l-1})+1}\right)\\
&\quad= \sum_{\beta_{l-1}=0}^{k-1} e_{1,1}\otimes s_{F_l(\beta,\beta_{l-1})+1}s^*_{F_l(\beta,\beta_{l-1})+1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\varphi^l(e_{1,1} \otimes 1) &= \sum_{\beta_0,\ldots,\beta_{l-2}=0}^{k-1} \varphi(e_{F_{l-1}(\beta)+f(l-1)+1,F_{l-1}(\beta)+f(l-1)+1} \otimes 1) \\
&= \sum_{\beta_0,\ldots,\beta_{l-1}}^{k-1} e_{1,1} \otimes s_{F_l(\beta,\beta_{l-1})+1} s^*_{F_l(\beta,\beta_{l-1})+1} \\
&= e_{1,1} \otimes 1
\end{aligned}$$

because $F_l(\beta, \beta_{l-1}) + 1$ maps bijectively onto $\{1, \ldots, n\}$.

Now suppose instead that $i > 1$ so that $p > 0$ and therefore $d = l-1-p < l-1$. Then Lemma 2.17 implies that

$$\varphi^{l-1}(e_{1,i} \otimes 1) = \sum_{\beta_0,\ldots,\beta_{l-2}=0}^{k-1} e_{F_{l-1}(\beta)+f(l-1)+1,A_j(l-1)+f(p-1)+1} \otimes s^*_{S_{j,l-1}(\beta)+1}.$$

We aim to calculate

$$\begin{aligned}
&\varphi(e_{F_{l-1}(\beta)+f(l-1)+1,A_j(l-1)+f(p-1)+1} \otimes s^*_{S_{j,l-1}(\beta)+1}) \\
&\quad = \varphi(e_{1,F_{l-1}(\beta)+f(l-1)+1} \otimes 1)^* \varphi(e_{1,1} \otimes s^*_{S_{j,l-1}(\beta)+1}) \varphi(e_{1,A_j(l-1)+f(p-1)+1} \otimes 1)
\end{aligned}$$

for fixed, but arbitrary β.

We have already calculated all three of these terms: $\varphi(e_{1,A_j(l-1)+f(p-1)+1} \otimes 1)$ was calculated in (2.20), $\varphi(e_{1,1} \otimes s^*_{S_{j,l-1}(\beta)+1})$ in (2.21), and the adjoint of $\varphi(e_{1,F_{l-1}(\beta)+f(l-1)+1} \otimes 1)^*$ in (2.25). The expression in (2.20) is accurate for $\varphi(e_{1,A_j(l-1)+f(p-1)+1} \otimes 1)$ because $A_j(l-1) < k^{p-1}$ (recall that $j < k^p$), so

$$A_j(l-1) + f(p-1) + 1 \leq k^p - 1 + f(p-1) + 1 = f(p) \leq f(l-1).$$

Multiplying the adjoint of (2.25) and (2.21) gives us

$$
\begin{aligned}
&\varphi(e_{F_{l-1}(\beta)+f(l-1)+1,1} \otimes s^*_{S_{j,l-1}(\beta)+1}) \\
&\quad = \varphi(e_{1,F_{l-1}(\beta)+f(l-1)+1} \otimes 1)^* \varphi(e_{1,1} \otimes s_{S_{j,l-1}(\beta)+1})^* \\
&\quad = \left(\sum_{\beta_{l-1}=0}^{k-1} e_{1,\beta_{l-1}+f(1)+1} \otimes s_{F_l(\beta,\beta_{l-1})+1} \right) \\
&\qquad \left(\sum_{\beta_{l-1}=0}^{k-1} e_{\beta_{l-1}+f(1)+1,a_{l-(l-1)-1}+f(1)+1} \otimes s^*_{S_{j,l}(\beta,\beta_{l-1})+1} \right) \\
&\quad = \sum_{\beta_{l-1}=0}^{k-1} e_{1,a_0+f(1)+1} \otimes s_{F_l(\beta,\beta_{l-1})+1} s^*_{F_l(\beta,\beta_{l-1})+1},
\end{aligned}
$$

where we use the fact that $S_{j,l}(\beta, \beta_{l-1}) = F_l(\beta, \beta_{l-1})$ in the last step. Right-multiplying the above product by (2.20) yields

$$
\begin{aligned}
&\varphi(e_{F_{l-1}(\beta)+f(l-1)+1,A_j(l-1)+f(p-1)+1} \otimes s^*_{S_{j,l-1}(\beta)+1}) \\
&\quad = \varphi(e_{F_{l-1}(\beta)+f(l-1)+1,1} \otimes s^*_{S_{j,l-1}(\beta)+1}) \varphi(e_{1,A_j(l-1)+f(p-1)+1} \otimes 1) \\
&\quad = \left(\sum_{\beta_{l-1}=0}^{k-1} e_{1,a_0+f(1)+1} \otimes s_{F_l(\beta,\beta_{l-1})+1} s^*_{F_l(\beta,\beta_{l-1})+1} \right) \\
&\qquad \left(\sum_{\beta_{l-1}=0}^{k-1} e_{\beta_{l-1}+f(1)+1,k\cdot A_j(l-1)+f(l-1-d)+\beta_m+1} \otimes s_{F_l(\beta,\beta_{l-1})+1} s^*_{F_l(\beta,\beta_{l-1})+1} \right) \\
&\quad = \sum_{\beta_{l-1}=0}^{k-1} e_{1,k\cdot A_j(l-1)+f(p)+a_0+1} \otimes s_{F_l(\beta,\beta_{l-1})+1} s^*_{F_l(\beta,\beta_{l-1})+1}
\end{aligned}
$$

Since $k \cdot A_j(l-1) + a_0 = A_j(l) = j$ and $i = 1 + f(p) + j$, the above sum can be simplified to

$$
\varphi(e_{F_{l-1}(\beta)+f(l-1)+1,A_j(l-1)+f(p-1)+1} \otimes s^*_{S_{j,l-1}(\beta)+1}) = \sum_{\beta_m=0}^{k-1} e_{1,i} \otimes s_{F_l(\beta,\beta_{l-1})+1} s^*_{F_l(\beta,\beta_{l-1})+1}.
$$

Hence

$$
\begin{aligned}
\varphi^l(e_{1,i}\otimes 1) &= \sum_{\beta_0,\dots,\beta_{l-2}=0}^{k-1} \varphi(e_{F_{l-1}(\beta)+f(l-1)+1,A_j(l-1)+f(p-1)+1}\otimes s^*_{S_{j,l-1}(\beta)+1}) \\
&= \sum_{\beta_0,\dots,\beta_{l-1}=0}^{k-1} e_{1,i}\otimes s_{F_l(\beta,\beta_{l-1})+1}s^*_{F_l(\beta,\beta_{l-1})+1} \\
&= e_{1,i}\otimes 1
\end{aligned}
$$

as $F_l(\beta,\beta_{l-1})+1$ maps bijectively onto $\{1,\dots,n\}$. This proves the first claim.

Now let $j\in\{0,\dots,n-1\}$ be as in Notation 2.13. Lemma 2.3 implies that

$$
\varphi(e_{1,1}\otimes s_{j+1}) = \sum_{\beta_0,\dots,\beta_{l-2}=0}^{k-1} e_{A_j(l-1)+f(l-1)+1,F_{l-1}(\beta)+f(l-1)+1}\otimes s_{S_{j,l-1}(\beta)+1}.
$$

We set out to calculate

$$
\begin{aligned}
&\varphi(e_{A_j(l-1)+f(l-1)+1,F_{l-1}(\beta)+f(l-1)+1}\otimes s_{S_{j,l-1}(\beta)+1}) \\
&\quad = \varphi(e_{1,A_j(l-1)+f(l-1)+1}\otimes 1)^*\varphi(e_{1,1}\otimes s_{S_{j,l-1}(\beta)+1})\varphi(e_{1,F_{l-1}(\beta)+f(l-1)+1}\otimes 1)
\end{aligned}
$$

for fixed, but arbitrary β.

We have already calculated $\varphi(e_{1,1}\otimes s_{S_{j,l-1}(\beta)+1})$ and $\varphi(e_{1,F_{l-1}(\beta)+f(l-1)+1}\otimes 1)$ in (2.21) and (2.25), respectively, so we need only calculate $\varphi(e_{1,A_j(l-1)+f(l-1)+1}\otimes 1)^*$. Clearly $A_j(l-1)+f(l-1)+1\geq f(l-1+1)$, so we use equation (2.2):

$$
\begin{aligned}
\varphi(e_{1,A_j(l-1)+f(l-1)+1}\otimes 1)^* &= \sum_{\beta_{l-1}=0}^{k-1} e_{1,\beta_{l-1}+f(1)+1}\otimes s_{\sigma(A_j(l-1)+f(l-1)+1)+\beta_{l-1}+1}. \\
&= \sum_{\beta_{l-1}=0}^{k-1} e_{1,\beta_{l-1}+f(1)+1}\otimes s_{k\cdot A_j(l-1)+\beta_{l-1}+1}.
\end{aligned} \tag{2.26}
$$

Multiplying (2.26) and (2.21) together yields

$$
\begin{aligned}
\varphi(e_{A_j(l-1)+f(l-1)+1,1} \otimes s_{S_{j,l-1}(\beta)}) &= \varphi(e_{1,A_j(l-1)+f(l-1)+1} \otimes 1)^* \varphi(e_{1,1} \otimes s_{S_{j,l-1}(\beta)+1}) \\
&= \left(\sum_{\beta_{l-1}=0}^{k-1} e_{1,\beta_{l-1}+f(1)+1} \otimes s_{k \cdot A_j(l-1)+\beta_{l-1}+1} \right) \\
&\quad \left(\sum_{\beta_{l-1}=0}^{k-1} e_{a_{l-(l-1)-1}+f(1)+1,\beta_{l-1}+f(1)+1} \otimes s_{S_{j,l}(\beta,\beta_{l-1})+1} \right) \\
&= \sum_{\beta_{l-1}=0}^{k-1} e_{1,\beta l-1+f(1)+1} \otimes s_{k \cdot A_j(l-1)+a_0+1} s_{S_{j,l}(\beta,\beta_{l-1})+1} \\
&= \sum_{\beta_{l-1}=0}^{k-1} e_{1,\beta l-1+f(1)+1} \otimes s_{j+1} s_{F_l(\beta,\beta_{l-1})+1}.
\end{aligned}
$$

Right-multiplying the above product by (2.25) gives us

$$
\begin{aligned}
&\varphi(e_{A_j(l-1)+f(l-1)+1,F_{l-1}(\beta)+f(l-1)+1} \otimes s_{S_{j,l-1}(\beta)+1}) \\
&= \varphi(e_{A_j(l-1)+f(l-1)+1,1} \otimes s_{S_{j,l-1}(\beta)+1}) \varphi(e_{1,F_{l-1}(\beta)+f(l-1)+1} \otimes 1) \\
&= \left(\sum_{\beta_{l-1}=0}^{k-1} e_{1,\beta_{l-1}+f(1)+1} \otimes s_{j+1} s_{F_l(\beta,\beta_{l-1})+1} \right) \\
&\quad \left(\sum_{\beta_{l-1}=0}^{k-1} e_{\beta_{l-1}+f(1)+1,1} \otimes s^*_{F_l(\beta,\beta_{l-1})+1} \right) \\
&= \sum_{\beta_{l-1}=0}^{k-1} e_{1,1} \otimes s_{j+1} s_{F_l(\beta,\beta_{l-1})+1} s^*_{F_l(\beta,\beta_{l-1})+1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\varphi^l(e_{1,1} \otimes s_{j+1}) &= \sum_{\beta_0,\ldots,\beta_{l-2}=0}^{k-1} \varphi(e_{A_j(l-1)+f(l-1)+1,F_{l-1}(\beta)+f(l-1)+1} \otimes s_{S_{j,l-1}(\beta)+1}) \\
&= \sum_{\beta_0,\ldots,\beta_{l-1}=0}^{k-1} e_{1,1} \otimes s_{j+1} s_{F_l(\beta,\beta_{l-1})+1} s^*_{F_l(\beta,\beta_{l-1})+1} \\
&= e_{1,1} \otimes s_{j+1}.
\end{aligned}
$$

This completes the proof. □

Corollary 2.19. Let k, l, and n be as in Notation 2.7 and let φ be as in Theorem 2.10. Then $\mathrm{id}_{M_{k-1}} \otimes \varphi$ is an automorphism of $M_{n-1} \otimes \mathcal{O}_n$ such that $\varphi_*([1]) = k \cdot [1]$ and $\varphi^l = \mathrm{id}_{M_{n-1} \otimes \mathcal{O}_n}$.

Proof. This follows directly from the calculation $(k-1)f(l) = k^l - 1 = n - 1$. $\square$

2.4 Connection between Theorems 2.4 and 2.10

We shall now show how Theorems 2.4 and 2.10 relate, and specifically how to derive the latter from the former.

Lemma 2.20. Let k, l, and n be as in Notation 2.7. Set

$$V_{j+1} = \sum_{\beta=0}^{k-1} e_{\lambda(j)+f(1)+1,\beta+f(1)+1} \otimes s_{\mu(j)+\beta+1}$$

for all $j \in \{0, \ldots, n-1\}$. Then there exists a unique isomorphism $\psi_0 \colon M_1 \otimes \mathcal{O}_n \to M_k \otimes \mathcal{O}_n$ such that $\psi_0(e_{1,1} \otimes s_{j+1}) = V_{j+1}$ for all $j \in \{0, \ldots, n-1\}$, where we identify $M_k \otimes \mathcal{O}_n$ with the diagonal $(k \times k)$-matrix in $M_{f(l)} \otimes \mathcal{O}_n$ whose top-left corner is in the 2nd row and 2nd column.

Proof. Note that these V_{j+1} are the same as in Theorem 2.10, and that any homomorphism $M_1 \otimes \mathcal{O}_n \to M_k \otimes \mathcal{O}_n$ must send $e_{1,1} \otimes 1$ to

$$U_1 = \sum_{\beta=0}^{k-1} = e_{\beta+f(1)+1,\beta+f(1)+1} \otimes 1.$$

The fact there existence of a unique homomorphism ψ_0 such that $\psi_0(e_{1,1} \otimes s_{j+1}) = V_{j+1}$ for all j then follows from the proof of Theorem 2.10.

Injectivity is again a consequence of $M_1 \otimes \mathcal{O}_n$ being simple and ψ_0 being nonzero. Recall that, for any $q \in \{0, \ldots, k-1\}$,

$$\sum_{r=0}^{k^{l-1}-1} V_{qk^{l-1}+r+1} V_{qk^{l-1}+r+1}^* = e_{q+2,q+2} \otimes 1.$$

This comes from (2.10) and was used in the calculation immediately following (2.10). Similarly, we have

$$\sum_{r=0}^{k^{l-1}-1} V_{r+1}V_{qk^{l-1}+r+1}^* = e_{q+2,q+2} \otimes 1 = \sum_{r=0}^{k^{l-1}-1} \left(\sum_{\beta=0}^{k-1} e_{f(1)+1,\beta+f(1)+1} \otimes s_{kr+\beta+1} \right) \left(\sum_{\beta=0}^{k-1} e_{\beta+f(1)+1,q+\beta+f(1)+1} \otimes s_{kr+\beta+1}^* \right)$$
$$= \sum_{r=0}^{k^{l-1}-1} \sum_{\beta=0}^{k-1} e_{f(1)+1,q+f(1)+1} \otimes s_{kr+\beta+1} s_{kr+\beta+1}^*$$
$$= \sum_{i=1}^{n} e_{2,q+2} \otimes s_i s_i^* = e_{2,q+2} \otimes 1$$

for all $q \in \{0, \ldots, k-1\}$. We now need only show that $e_{2,2} \otimes s_i$ is in the image of ψ_0 for all $i \in \{1, \ldots, n\}$. Let $i \in \{1, \ldots, n\}$. Then there exists a unique $r \in \{0, \ldots, k^{l-1}-1\}$ and $a \in \{0, \ldots, k-1\}$ such that $i - 1 = rk + a$. Therefore

$$V_{r+1}(e_{a+2,2} \otimes 1) = \left(\sum_{\beta=0}^{k-1} e_{2,\beta+2} \otimes s_{kr+\beta+1} \right) (e_{a+2,2} \otimes 1) = e_{2,2} \otimes s_{kr+a+1} = e_{2,2} \otimes s_i.$$

Since $e_{a+2,2} \otimes 1$ is also in the image of ψ_0, this demonstrates that ψ_0 is surjective. $\square$

Theorem 2.21. Let k, l, and n be as in Notation 2.7, let ψ_0 be as in Lemma 2.20, and let φ be as in Theorem 2.10. For each $p \in \{1, \ldots, l-2\}$, define an isomorphism $\psi_p \colon M_{k^p} \otimes \mathcal{O}_n \to M_{k^{p+1}} \otimes \mathcal{O}_n$ by $\psi_p = \mathrm{id}_{M_{k^p}} \otimes \psi_0$ and identify the matrix algebra $M_{k^p} \otimes \mathcal{O}_n$ with the diagonal $(k^p \times k^p)$-matrix in $M_{f(l)} \otimes \mathcal{O}_n$ whose top left corner is located in the $(1 + f(p))$-th row and $(1 + f(p))$-th column. Then $\varphi(e_{i,j} \otimes v) = \psi_p(e_{i,j} \otimes v)$ for all $i, j \in \{1 + f(p), \ldots, f(p+1)\}$ and $v \in \mathcal{O}_n$.

Furthermore, for any $v \in \mathcal{O}_n$ and $m \in \{1, \ldots, l-1\}$, we have

$$\varphi^m(e_{1,1} \otimes v) = \big(\psi_{m-1} \circ \cdots \psi_1 \circ \psi_0\big)\big(e_{1,1} \otimes v\big).$$

Proof. First, observe that $\varphi(e_{1,1} \otimes s_{j+1}) = \psi_0(e_{1,1} \otimes s_{j+1})$ for all $j \in \{0, \ldots, n-1\}$ by the definition of ψ_0. Thus φ restricts to ψ_0 on $M_1 \otimes \mathcal{O}_n$.

Now let $p \in \{1, \ldots, l-2\}$ and let $i, j \in \{0, \ldots, k^p - 1\}$. Since $\psi_p = \mathrm{id}_{M_{k^p}} \otimes \psi_0$, we have

$$\begin{aligned}
\psi_p(e_{1,1+f(p)+i} \otimes 1) &= e_{1,1+f(p)+i} \otimes \left(\sum_{\alpha=0}^{k-1} e_{\alpha+f(1)+1,\alpha+f(1)+1} \otimes 1 \right) \\
&= \left(\sum_{\alpha=0}^{k-1} e_{\alpha+f(p)+1,\alpha+f(p)+1} \otimes 1 \right) \left(\sum_{\beta=0}^{k-1} e_{\beta+f(p)+1,k(i-1)+f(p)+1} \otimes 1 \right) \\
&= \sum_{\alpha=0}^{k-1} e_{\alpha+f(p)+1,k(i-1)+\alpha+f(p)+1} \otimes 1 \\
&= \varphi(e_{1,1+f(p)+i} \otimes 1).
\end{aligned}$$

Likewise, $\psi_p(e_{1,1+f(p)+j} \otimes 1) = \varphi(e_{1,1+f(p)+j} \otimes 1)$. Then, for all $v \in \mathcal{O}_n$, we have

$$\begin{aligned}
\varphi(e_{i,j} \otimes v) &= \varphi(e_{1,i} \otimes 1)^* \varphi(e_{1,1} \otimes v) \varphi(e_{1,j} \otimes 1) \\
&= \psi_p(e_{1,j} \otimes 1)^* \psi_0(e_{1,1} \otimes v) \psi_p(e_{1,i} \otimes 1) \\
&= \psi_p(e_{i,j} \otimes v).
\end{aligned}$$

For the second assertion, let $v \in \mathcal{O}_n$. We proceed by induction. The base case follows from the first assertion of the lemma, so let $m \in \{0, \ldots, l-2\}$ and suppose that

$$\varphi^m(e_{1,1} \otimes v) = (\psi_{m-1} \circ \cdots \psi_1 \circ \psi_0)(e_{1,1} \otimes v).$$

Since ψ_p maps onto the domain of ψ_{p+1} for all $p \in \{0, \ldots, l-3\}$, we know that the right-hand expression is an element of $M_{k^m} \otimes \mathcal{O}_n$. We know that φ restricts to ψ_m on $M_{k^m} \otimes \mathcal{O}_n$ from the first assertion, so

$$\varphi^{m+1}(e_{1,1} \otimes v) = (\psi_m \circ \varphi^m)(e_{1,1} \otimes v) = (\psi_m \circ \psi_{m-1} \circ \cdots \psi_1 \circ \psi_0)(e_{1,1} \otimes v).$$

This completes the proof. $\square$

Corollary 2.22. Let k, l, and n be as in Notation 2.7, let $\psi_0, \ldots, \psi_{l-2}$ be as in Theorem 2.21, and let φ be as in Theorem 2.10. Define $\psi_{l-1}\colon M_{k^{l-1}} \otimes \mathcal{O}_n \to M_1 \otimes \mathcal{O}_n$ by

$$\psi_{l-1} = \psi_0^{-1} \circ \psi_1^{-1} \circ \cdots \circ \psi_{l-2}^{-1},$$

as in the proof of Theorem 2.4. Then $\varphi(e_{i,j} \otimes v) = \psi_{l-1}(e_{i,j} \otimes v)$ for all $i, j \in \{1 + f(l-1), \ldots, f(l)\}$ and $v \in \mathcal{O}_n$.

Proof. Let $i, j \in \{1 + f(l-1), \ldots, f(l)\}$ and let $v \in \mathcal{O}_n$. Then

$$\psi_{l-1}(e_{i,j} \otimes v) = (\varphi^l \circ \psi_{l-1})(e_{i,j} \otimes v) = (\varphi \circ \psi_{l-2} \circ \cdots \circ \psi_0 \circ \psi_{l-1})(e_{i,j} \otimes v) = \varphi(e_{i,j} \otimes v)$$

because φ has order l. $\square$

Theorem 2.21 and Corollary 2.22 together imply that $\varphi(e_{i,i} \otimes 1) = \psi_p(e_{i,i} \otimes 1)$ for all $i \in \{1 + f(p), \ldots, f(p+1)\}$ and $\varphi(e_{1,1} \otimes s_{j+1}) = \psi_0(e_{1,1} \otimes s_{j+1})$ for all $j \in \{0, \ldots, n-1\}$. This is precisely how we defined φ in the proof of Theorem 2.4, showing that Theorem 2.10 is a specific instance of Theorem 2.4.

In fact, because φ restricts to ψ_{l-1} on $M_{k^{l-1}} \otimes \mathcal{O}_n$, we have the curious relation

$$\begin{aligned}
(\psi_0^{-1} \circ \psi_1^{-1} \circ \cdots \circ \psi_{l-2}^{-1})(e_{i,j} \otimes v) &= \psi_{l-1}(e_{i,j} \otimes v) \\
&= \varphi(e_{i,j} \otimes v) \\
&= \varphi(e_{1,i} \otimes 1)^* \psi_0(e_{1,1} \otimes v) \varphi(e_{1,j} \otimes 1) \qquad (\dagger)
\end{aligned}$$

for all $i, j \in \{1 + f(l-1), \ldots, f(l)\}$ and $v \in \mathcal{O}_n$. We have effectively shown that (†) holds for any automorphism derived from Theorem 2.4 that has order l, although (†) is not sufficient to imply φ has order l. Further work will have to be done to determine what properties an automorphism derived from Theorem 2.4 require so that it has order l, and if and how these properties can be realized.

References

[1] G. Abrams, P. N. Ánh, and E. Pardo. Isomorphisms between Leavitt algebras and their matrix rings. *J. Reine Angew. Math.*, 624:103–132, 2008.

[2] P. Ara, K. R. Goodearl, and E. Pardo. K_0 of purely infinite simple regular rings. *K-Theory*, 26(1):69–100, 2002.

[3] Bruce Blackadar. *K-Theory for Operator Algebras*. Cambridge University Press, 2 edition, 1998.

[4] Joachim Cuntz. Simple C^*-algebras generated by isometries. *Comm. Math. Phys.*, 57(2):173–185, 1977.

[5] Joachim Cuntz. K-theory for certain C^*-algebras. *Ann. of Math. (2)*, 113(1):181–197, 1981.

[6] Takeshi Katsura. A construction of actions on Kirchberg algebras which induce given actions on their K-groups. preprint (arXiv: math.OA/0608093), 2007.

[7] Terry A. Loring. *Lifting solutions to perturbing problems in C^*-algebras*, volume 8 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1997.

[8] Terry A. Loring. C^*-algebra relations. *Math. Scand.*, 107(1):43–72, 2010.

[9] N.E. Wegge-Olsen. *K-Theory and C*-Algebras*. Oxford University Press, 1993.