Symmetric Functions

& the Character Table of S_n

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Introduction

This document is an exposition on the classical theory of symmetric functions and how it can be used to obtain the character table of S_n. Loosely speaking a symmetric function is a polynomial in some list of variables $\{x_1, x_2, \dots\}$ (could be finite or infinite) which is invariant under permutation of the indices (the natural action of S_n).

We begin with giving a whirlwind review of the representation theory of finite groups and character theory in Section 1. We are only ever working over $\mathbb{C}$. In this section we state many well known theorems from representation theory, and give a discussion on the inner product on characters. Section 1 is filled with examples mostly pertaining to the symmetric group as this is the primary focus. We build up to creating character tables for the symmetric group, and then move on to studying symmetric functions independently in Section 2. We give a detailed discussion on Λ, the algebra of symmetric functions, and all the various families of symmetric functions which live there. We also include a detailed discussion on Schur functions, which are the most important basis for Λ, as they will turn out to correspond to irreducible characters of the symmetric group. Section 2 also develops a complex inner product on Λ, and discusses consequent orthogonality relations for some of our bases. Finally at the end of Section 2 we have the tools to create Kostka matrices. These matrices represent a transition map between two different bases of symmetric functions, and play a key role in obtaining the character table of the symmetric group.

The two topics converge in Section 3, which is a construction of a correspondence between the algebra of symmetric functions Λ and the space R of class functions on S_n. It is well known that the number of irreducible characters of a finite group G equals the number of conjugacy classes in G, and in the symmetric group S_n this is further equal to the number of partitions of n. The characteristic map will allow us to realize exactly which partition a given irreducible character χ should correspond to, via the realization that the Schur symmetric functions represent irreducible characters under the characteristic map. It is at this point that we can recover our character tables of the symmetric group from Section 1 by realizing the character table as a transition map between certain bases of Λ. This is Frobenius' theorem proved originally in 1900 [**Fro00**]. Remarkably, the proof uses hardly any group/representation theory whatsoever! We also include some basic examples of code in Sage which produce the character tables and other transition maps. Most of the results from Section 1 can be found in [**JL93**], and likewise for Sections 2 and 3 in [**Mac79**] and [**Sag01**].

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1 Representations, $\mathbb{C}G$-modules, and Characters

1.1 Representations and $\mathbb{C}G$-modules

This section lists some of the definitions that we will use throughout the rest of the document, and state some important results from character theory. We will always take our group G to be S_n for some n, and our field $\mathbb{F}$ will be $\mathbb{C}$, so all of the baby examples from this section will use the symmetric group, and use $\mathbb{C}$ as the ground field.

Definition 1.1.1: Let G be a finite group. A *representation* of G is a homomorphism $\varphi : G \to GL_n(\mathbb{C})$ for some n.

Example: For $V = \mathbb{C}^n$, and $G = S_n$, there is always a homomorphism $\rho : G \to GL(V)$ such that $g \mapsto \mathcal{P}_g$, where $\mathcal{P}_g$ is the corresponding permutation matrix, i.e., the matrix sending the basis vector $e_i \mapsto e_{g(i)}$ for $1 \leq i \leq n$. For example, when $n = 4$, some examples of permutations matrices are

$$(123) \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2314) \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (13) \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Definition 1.1.2: Let $\rho : G \to GL_n(\mathbb{C})$ and $\sigma : G \to GL_m(\mathbb{C})$ be two representations of G. We say that ρ is *equivalent* to σ if $n = m$ and there exists an invertible $n \times n$ matrix T such that for all $g \in G$ we have

$$\sigma(g) = T^{-1}\rho(g)T.$$

It is clear that equivalence of representations is an equivalence relation. Note that for every group G, there is a *trivial representation* $\rho : G \to GL_1(\mathbb{C}) = \mathbb{C}^\times$, $g \mapsto 1$.

Definition 1.1.3: Let G be a group and V a vector space over $\mathbb{C}$. The group G is said to act *linearly* on V if it acts on V in the usual way, and additionally:

$$g \cdot (\lambda v) = \lambda(g \cdot v) \tag{1}$$

$$g \cdot (v + w) = g \cdot v + g \cdot w \tag{2}$$

The notion of a linear group action gives rise to an alternative way to obtain representation of a finite group G: if we are given a finite dimensional vector space V over $\mathbb{C}$, if we define a linear group action on V and choose a basis, we obtain a matrix representation $[g]$ for each $g \in G$:

$$g \cdot v = [g] \cdot v, \quad v \in V, g \in G.$$

A vector space with this extra structure of a linear group action is usually called a $\mathbb{C}G$-module, but we will not give that full definition here. Note that given a representation of a finite group G, we have a corresponding $\mathbb{C}G$-module, and conversely by choosing a basis, given an $\mathbb{C}G$-module we obtain a representation of G. This correspondence is important, and from now on we will mostly use the language of $\mathbb{C}G$-modules.

Definition 1.1.4: Let V be a $\mathbb{C}G$-module. A subspace $W \subseteq V$ is said to be a *submodule* of V if $g \cdot W \subseteq W$, i.e., $g \cdot w \in W$ for all $g \in G$, $w \in W$.

Example: We continue on in a similar way to our previous example with symmetric groups. The symmetric group S_n acts on the left on the set $\{1, 2, \dots, n\}$ as always. Let V be the free complex vector space spanned by formal elements $\{v_1, v_2, \dots v_n\}$. Then S_n acts on $\{v_1, \dots, v_n\}$ by $g \cdot v_i = v_{g(i)}$, and if we linearize this action, V is turned into an $\mathbb{C}S_n$-module, called the natural permutation module. Observe that $M = \text{span}\{v_1 + v_2 + \dots + v_n\}$ is a 1-dimensional submodule of V, since if any $g \in G$ acts on $v_1 + v_2 + \dots + v_n$ by permuting the basis in some way we have $g \cdot (v_1 + v_2 + \dots + v_n) = g(v_1) + g(v_2) + \dots + g(v_n) = v_1 + v_2 + \dots + v_n \in M$.

Definition 1.1.5: A $\mathbb{C}G$-module V is *irreducible* if it has no non-trivial submodules, i.e, no submodules other than $\{0\}$ and V itself.

Example: Still following our previous example, since we've found a non-trivial submodule $M \subset V$, this means that V is not irreducible. Naturally M itself is irreducible as a $\mathbb{C}G$-module, since it is 1-dimensional.

Example: Let G be a finite group and let $V = \mathbb{C}G$, the group algebra of G. Then $\mathbb{C}G$ is naturally a $\mathbb{C}G$-module, where the action of G is given by left multiplication. This is called the left regular module of G and it has dimension $|G|$.

Definition 1.1.6: Let V and W be $\mathbb{C}G$-modules and let $\varphi : V \to W$ be a linear map. Then φ is a *$\mathbb{C}G$-module homomorphism* if

$$\varphi(g \cdot v) = g \cdot \varphi(v), \quad g \in G.$$

Note that this agrees with the usual definition of an R-module homomorphism; we just need our function φ to commute past elements of R (in this case, elements of G since we have $R = \mathbb{C}G$ and φ is already linear).

As expected, the kernel and image of an $\mathbb{C}G$-module homomorphism $\varphi : V \to W$ are submodules of V and W respectively.

Theorem 1.1.7 (Mashke's Theorem): Let G be a finite group and let V be a $\mathbb{C}G$-module. If $U \subset V$ is is a submodule, then there exists a submodule $W \subset V$ such that $V = U \oplus W$.

An important consequence of Mashke's Theorem is that every nonzero $\mathbb{C}G$-module for a finite group G is completely reducible. That is, any $\mathbb{C}G$-module V decomposes as $U_1 \oplus \dots \oplus U_n$ where each U_i is an irreducible submodule of V.

Example: The permutation $\mathbb{C}S_n$-module V arising from the action of S_n on $\{v_1, v_2, \dots, v_n\}$ (see the example following Definition 1.1.4) decomposes as $V = M \oplus L$ where $M = \text{span}(v_1 + v_2 + \dots + v_n)$, and L is another irreducible submodule consisting of "trace zero" vectors. Irreducibility of L follows as $\langle \chi_L, \chi_L \rangle = 1$ (see 1.3 for this inner product and explanation.)

1.2 Characters

As previously stated, the main purpose of this project is to show how symmetric functions can be used to yield the character table of S_n. In this section we define characters, and we use results from character theory to make some character tables for S_n for small n.

Definition 1.2.1: Let V be a $\mathbb{C}G$-module, and let φ be the corresponding representation. The *character* of V is the function $\chi : G \to \mathbb{C}$ defined by

$$\chi(g) = \operatorname{tr}[\varphi(g)], \quad g \in G.$$

The character of V is a class function: it is a function that is constant on conjugacy classes of G. This is due to the property of trace: $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. For any matrices $A, B \in GL_n(\mathbb{C})$ we have

$$\operatorname{tr}(ABA^{-1}) = \operatorname{tr}(BAA^{-1}) = \operatorname{tr}(B).$$

In particular the above argument implies that if $g, h \in G$ and g is conjugate to h, we have that $\chi(g) = \chi(h)$. This is important for the symmetric group since conjugacy classes in the symmetric group are determined by cycle type, which motivates the use of partitions. We say that χ is an irreducible character of G if χ is the character of an irreducible $\mathbb{C}G$-module, otherwise it is said to be reducible.

Note that if V, W are two isomorphic $\mathbb{C}G$-modules, there are basis B_1, B_2 of V and W respectively such that $\left[g\right]_{B_1} = \left[g\right]_{B_2}$ for all $g \in G$. Then as a consequence, if V, W are isomorphic $\mathbb{C}G$-modules, then the character of V equals the character of W.

Definition 1.2.2: Let V be a $\mathbb{C}G$-module with character χ. The *degree* of χ is defined as $\chi(1)$, which is equal to $\dim_{\mathbb{C}} V$. A character of degree 1 is said to be a linear character.

Note that linear characters are irreducible.

Example: Consider the trivial representation of G over $\mathbb{C}$. Every $g \in G$ is sent to $1 \in \mathbb{C}^{\times}$, which is trivially just a 1×1 matrix. Hence the trace of every matrix is 1, so the character of the corresponding module is 1 on every conjugacy class of G. This is called the trivial character.

Example: There always exists a homomorphism $S_n \to \mathbb{C}^{\times}$ with $g \mapsto \operatorname{sgn}(g) = \pm 1$. This is called the sign representation of S_n, and the corresponding character is linear as well (it is degree 1.)

Let V be a $\mathbb{C}G$-module with $V = U_1 \oplus \cdots \oplus U_m$ where each U_i is an irreducible submodule. The corresponding matrices for G take a block diagonal form with respect to each submodule, and so taking the trace of each matrix one can see that the character of V is the sum of the characters of $U_1, \ldots, U_n$.

Example: Suppose S_n acts on a finite set X with $|X| = m$. Let V be the $\mathbb{C}$-vector space spanned by formal elements $\{v_i\}, i \in X$. Then V is a $\mathbb{C}G$-module with basis $v_1, v_2, \ldots, v_m$ such that for all $g \in S_n$ we have $g \cdot v_i = v_{g(i)}$. Let φ be the representation associated to this module. Then the permutation matrix corresponding to g has diagonal entry 0 unless $g(i) = i$, in which case it is 1 (see the example following Definition 1.1.1 for some permutation matrices). Thus the trace of the matrix $\varphi(g)$ equals the number of fixed points in X when $g \in S_n$ acts on X, so the character π of this module is

$$\pi(g) = |\operatorname{fix}(g)|.$$

where $\operatorname{fix}(g)$ is the number of fixed points of g in X. For example, when S_n acts on $X = \{1, 2, \ldots, n\}$ the element $g = (1234) \in S_7$ has three fixed points, namely 5,6, and 7. The character π is called the permutation character of $G = S_n$. The permutation module associated to the usual action of S_n on $\{1, 2, \ldots, n\}$ is called the natural permutation module. For an example of a non-natural permutation module, see the end of Section 1.5.

Theorem 1.2.3: Let V be a permutation module for S_n. Then $\pi_V : S_n \to \mathbb{C}$, $g \mapsto |\operatorname{fix}(g)|$ is a character of S_n. Furthermore, the function $\chi : S_n \to \mathbb{C}$ sending $g \mapsto |\operatorname{fix}(g)| - 1$ is also a character of S_n.

The outline of the proof in the case of the natural permutation module is as is follows: By Mashke's Theorem, for the submodule $M = \operatorname{span}\{e_1, e_2, e_3\}$, there exists a submodule U such that $V = M \oplus U$. Observe that the character of M is the trivial character since U is isomorphic to the trivial module. Therefore by Theorem

1.2.2, if π is the character of V and χ is the character of U we have $\pi(g) = 1 + \chi(g)$, and since $\pi(g)$ is the number of fixed points we have that

$$\chi(g) = |\mathrm{fix}(g)| - 1$$

is the character of U.

1.3 Inner Products of Characters

We mentioned before that characters are constant on conjugacy classes. Functions constant on conjugacy classes are called class functions, and the space of all such functions $R(G)$ is a vector space: given two class functions $\chi, \psi : G \to \mathbb{C}$, addition and scalar multiplication is defined by

$$(\chi + \psi)(g) = \chi(g) + \psi(g)$$
$$\lambda \cdot \psi(g) = \lambda(\psi(g)).$$

In fact, $R(G)$ is a ring with pointwise multiplication $(\chi \cdot \psi)(g) = \chi(g)\psi(g)$. The space of class functions can be equipped with a complex inner product:

Definition 1.3.1: Let χ, ψ be class functions $\chi, \psi : G \to \mathbb{C}$. Define an inner product $\langle \cdot, \cdot \rangle$ on $R(G)$ by

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}.$$

One can check that the above is conjugate symmetric and sesquilinear. We now list some important results, the proofs for all of which can be found in Liebeck [**JL93**], our basic reference for character theory.

Theorem 1.3.2: Let $\chi_1, \dots, \chi_n$ be a complete set of non-isomorphic irreducible characters of G. Then

$$\sum_{i=1}^{n} (\chi_i(1))^2 = |G|.$$

The above theorem isn't too difficult to prove; it follows from the Artin-Wedderburn theorem applied to the semisimple algebra $\mathbb{C}G$. However the next theorem is much harder to prove: the proof requires properties of algebraic integers.

Theorem 1.3.3: Let χ be an irreducible character of G. Then $\chi(1)$ divides $|G|$.

One of the most important result that we will use very frequently is the following:
Theorem 1.3.4 (Schur's Lemma): Let U, V be non-isomorphic irreducible $\mathbb{C}G$-modules with characters χ and ψ respectively. Then

$$\langle \chi, \chi \rangle = 1,$$
$$\langle \chi, \psi \rangle = 0.$$

In other words, irreducible characters are orthonormal in $R(G)$. In particular, given a complete set of irreducible characters $\chi_1, \chi_2, \dots, \chi_n$ of G, we have $\langle \chi_i, \chi_j \rangle = \delta_{ij}$. This implies that $\chi_1, \chi_2, \dots, \chi_n$ are linearly independent in $R(G)$. Furthermore if ψ is any character then

$$\psi = \sum_{i=1}^{n} d_i \chi_i$$

where $d_i = \langle \psi, \chi_i \rangle$.

Corollary 1.3.5: Let ψ be a character of G. Then ψ is irreducible if and only if $\langle \psi, \psi \rangle = 1$.

Since any character is a sum of irreducibles, this is a direct corollary of Schur's Lemma.

Theorem 1.3.6: The number of irreducible characters of G is the number of conjugacy classes of G. Equivalently, the number of isomorphism classes of irreducible representations of G is equal to the number of conjugacy classes in G.

In $R(G)$, one has a basis $\{\delta_i\}$ consisting of functions δ_i called indicator functions. These are functions which take the value 1 on a single conjugacy class and zero on all other classes. These functions will be discussed in more detail later. It follows that $\dim R(G)$ is equal to the number of conjugacy classes of G.

Since the irreducible characters $\chi_1, \chi_2, \ldots, \chi_n$ are linearly independent elements of $R(G)$, if l is the number of conjugacy classes then we have $n \leq l$. One can further consider the dimension of $Z(\mathbb{C}G)$, the center of the group algebra of G over $\mathbb{C}$, to find that $\dim Z(\mathbb{C}G) = l$, and the above result follows after applying the Artin-Wedderburn theorem. Therefore the $\{\chi_i\}$ are also a basis for $R(G)$.

Note that characters are only closed on $\mathbb{N}$-linear combinations of irreducible characters, so the space of characters of G is not a vector space. Occasionally we will need to deal with *virtual* characters. A virtual character is defined as any $\mathbb{Z}$-linear combination of irreducible characters $\{\chi_i\}$.

1.4 Character Tables

Now that we know that the number of irreducible characters of a finite group G is equal to the number of conjugacy classes of G, we can begin to explore character tables, particularly for the symmetric group. We can use many of the tricks from this section to fill out character tables for small n (say, $n \leq 5$). By the end of Section 3, we will know how to obtain the character table of S_n for *any* n, and with hardly any of the tricks from this section whatsoever!

Definition 1.4.1: Let G be a finite group and let $\chi_1, \chi_2, \ldots, \chi_n$ be the irreducible characters of G and let $g_1, g_2, \ldots, g_n$ be representatives for the conjugacy classes in G. The *character table* of G is an $n \times n$ matrix whose ij-entry is $\chi_i(g_j)$ with $1 \leq i \leq n, 1 \leq j \leq n$.

In other words, the character table of G is an array whose columns are indexed by conjugacy classes and whose rows are indexed by irreducible characters. Note that the character table of G is always an invertible matrix, since we know that its rows (the irreducible characters) are linearly independent.
Example Let's make our first character table. Let $G = S_3$. Conjugacy classes in S_n are determined by cycle type, and we have three possible cycle types $1, (12), (123)$ in S_3. Therefore there are 3 irreducible characters. To connect to some previously stated results: this is reflected in the fact that $|S_3| = 6 = 1^2 + 1^2 + 2^2$. In particular we expect that the degrees of the three irreducible characters are 1,1, and 2. I claim that we already know what the corresponding modules are!

First, there is the trivial character χ_1, corresponding to the trivial representation, i.e., the map sending every $g \in G$ to the 1×1 identity matrix. This character is 1 on every class. The other degree 1 character corresponds to the sign homomorphism; so this character χ_{sgn} is just ± 1 depending on the sign of the permutation. Filling out these values in rows we have

S_3	1	(12)	(123)
χ_1	1	1	1
χ_{sgn}	1	-1	1
χ_3			

The last character χ_3 we have already computed - it is the permutation character χ_p. We know that the natural permutation module M for S_3 is 3-dimensional over $\mathbb{C}$, and its character was the number of fixed points. Recall that to account for the reducibility of this module we ended up finding a degree 2 character given by $|\text{fix}(g)| - 1$ (this was Theorem 1.2.3.) In fact, this character is irreducible.

Recall Burnside's Lemma: When G acts on a set X, the number of orbits is the average number of fixed points. Thus one has

$$\langle \chi_p, 1 \rangle = \#\text{ of orbits on X}$$

where in this case $X = \{1, 2, 3\}$. Since our action is transitive we have $\langle \chi_p, 1 \rangle = 1$. Furthermore we have $\langle \chi_p, \chi_p \rangle = \langle \chi_p \cdot \chi_p, 1 \rangle =$ the number of orbits on $X \times X$, which in this case equals 2, since we have two orbits on $X \times X$, being the diagonal (i.e., pairs $(a, a) \in X \times X$), and everything else. Hence we have $\langle \chi_p, \chi_p \rangle = 2$ and $\langle \chi_p, 1 \rangle = 1$, and it follows that $\langle \chi_p - 1, \chi_p - 1 \rangle = 1$, so $\chi_p - 1$ is irreducible (and we know it is a character by Theorem 1.2.3.)

The identity element fixes everything, (12) fixes 3, and (123) fixes nothing. Therefore the last three values are 2, 0, and -1. Thus we have

S_3	1	(12)	(123)
χ_1	1	1	1
χ_{sgn}	1	-1	1
χ_p	2	0	-1

and this is the character table for S_3. Once again, since it is important: S_n always has a irreducible degree $n - 1$ character, given by $|\text{fix}(g)| - 1$. The corresponding module is a great help in finding new irreducible characters, and will be used to construct character tables for S_4 and S_5 in the next section.

As previously discussed, we know that $\langle \chi_i, \chi_j \rangle = \delta_{ij}$ for two non-isomorphic irreducible characters. In terms of the rows of a character table, for any $r, s \in \{1, 2, \ldots, n\}$, this translates to

$$\sum_{i=1}^{n} \frac{\chi_r(g_i)\overline{\chi_s(g_i)}}{|C_G(g_i)|} = \delta_{rs}.$$

Another important result is that the columns are orthogonal as well:

Theorem 1.4.2: Let $\chi_1, \chi_2, \ldots, \chi_n$ be the irreducible characters of a finite group G, and let $g_1, g_2, \ldots, g_n$ be representatives for the conjugacy classes in G. Then for any $r, s \in \{1, 2, \ldots, n\}$, we have

$$\sum_{i=1}^{n} \chi_i(g_r) \cdot \overline{\chi_i(g_s)} = \delta_{rs}|C_G(g_r)|.$$

It turns out that in the symmetric group, all the characters are integers, so the conjugate in the above sums can be dropped in our case. One can verify using our example of the character table of S_3 on the previous page that these two relations hold. Note that when checking row orthogonality, you need to include the size of the centralizer in each term. We give a formula for this, since it will come up later and is actually very important.

When we write $g_\lambda \in S_n$, we mean element of cycle type $\lambda = (1^{m_1}, 2^{m_2}, 3^{m_3}, \ldots, n^{m_n})$, where in this notation each component i^{m_i} of the tuple λ represents m_i $i-$cycles in the disjoint cycle decomposition of g_λ. For example, if $\lambda = (2^2, 5)$, then g_λ consists of two 2-cycles and one 5-cycle, so then $g_\lambda \in S_9$ is an element of the form

$$g = (12)(34)(56789).$$

Lemma 1.4.3: Let

$$z_\lambda = \prod_{i\geq 1} i^{m_i} \cdot m_i!.$$

Then $z_\lambda = |C_{S_n}(g_\lambda)|$ where g_λ is an element of cycle type λ in S_n.

Proof. Let S_n act on itself by conjugation. Then given any $g_\lambda \in S_n$, the centralizer of g_λ is the stabilizer subgroup of the conjugation action. Therefore by the Orbit-Stabilizer theorem:

$$|G| = n! = |C_{S_n}(g_\lambda)| \cdot |\mathrm{Orb}(g_\lambda)|.$$

The size of the orbit of the conjugation action in the symmetric group is just the number of elements of cycle type λ. To count this, write $\lambda = (1^{m_1}, 2^{m_2}, 3^{m_3}, \ldots, n^{m_n})$ out in a tuple of length n making note of its cycles. To begin with, of course there are $n \cdot (n-1) \cdots = n!$ possible ways to arbitrarily label the entries in each cycle with $\{1, 2, \ldots, n\}$. Now, to count the actual number of elements in this conjugacy class we need to make sure we do not double count equivalent i-cycles, nor cycles of the same length in different positions. For a given i-cycle $\sigma = (12 \cdots i)$, there are i equivalent ways of writing σ, and if there are m_i of these cycles then there are i^{m_i} equivalent compositions of i-cycles. Furthermore, if there are m_i of these cycles, the cycles can be ordered in $m_i!$ ways which all yield the same overall element in S_n. Thus to avoid counting these extra elements we must divide $n!$ by i^{m_i} and $m_i!$ for each i to obtain the size:

$$|\mathrm{Orb}(g_\lambda)| = \frac{n!}{\prod\limits_{i\geq 1} i^{m_i} \cdot m_i!}$$

and then the result follows by the Orbit-Stabilizer theorem as previously mentioned.

□

1.5 Tensor Product Modules and Examples for S_4, S_5

One might hope that for two characters ψ, χ of a group G, that the function $(\psi \cdot \chi)(g) \equiv \psi(g) \cdot \chi(g)$ is a character as well. Indeed it is: if V, W are the corresponding $\mathbb{C}G$-modules for ψ and χ, $\psi \cdot \chi$ is a character for the tensor product module $V \otimes W$ (defined below). In particular, this will help us create new characters and fill out character tables easier. One especially useful application of this is taking powers of characters (although, we won't be needing this for our character tables).

Let G be a finite group and let V, W be $\mathbb{C}G$-modules with basis $\{v_i\}$, $1 \leq i \leq n$ and $\{w_j\}$, $1 \leq j \leq m$ respectively, for some n and m. We know that the set of all $v_i \otimes w_j$ is a basis for $V \otimes W$ as a vector space, and if we define an action of G on pure tensors and linearly extend, we turn $V \otimes W$ into a $\mathbb{C}G$-module.

Definition 1.5.1: For all $g \in G$, define

$$g(v_i \otimes w_j) \equiv g(v_i) \otimes g(w_j).$$

and in general define

$$g\bigg(\sum_{i,j}\lambda_{ij}(v_i\otimes w_j)\bigg)\equiv\sum_{i,j}\lambda_{ij}(g(v_i)\otimes g(w_j)).$$

It follows from the above that for any pure tensor $v\otimes w$ (not necessarily basis elements) we have $g(v\otimes w) = g(v)\otimes g(w)$. As a consequence, the action of G defined above turns $V\otimes W$ into a $\mathbb{C}G$-module.

Theorem 1.5.2: Let V, W be $\mathbb{C}G$-modules with characters ψ, χ respectively. Then the character of $V\otimes W$ is $\psi\cdot\chi$ (the pointwise product defined earlier).

One way to prove the above is by noting the following: if G is a finite group and V be a $\mathbb{C}G$-module, and φ the corresponding representation, then for all $g\in G$, there is a basis of V so that $\varphi(g)$ is diagonal. One can choose this basis for each module V, W, and then note that $g(v_i\otimes w_j) = g(v_i)\otimes g(w_j) = \lambda_i v_i\otimes\mu_j w_j = \lambda_i\mu_j(v_i\otimes w_j)$, and the result follows.

As a result, the product of any two characters is again a character (it is a character of the tensor product module). The final result we need to start making more characters is:

Corollary 1.5.3: Let χ be a character of G and let ψ be a linear character of G. Then $\chi\cdot\psi$ is irreducible if and only if χ is irreducible.

Example: We now have the basic tools to make the character tables for S_4 and S_5. Let's start with S_4. There are five conjugacy classes with representatives $1, (12), (123), (12)(34), (1234)$. Similar to how we computed the characters in S_3, we can immediately fill out three rows using the trivial character, sign character, and permutation character:

S_4	1	(12)	(123)	(1234)	(12)(34)
χ_1	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
χ_p	3	1	0	-1	-1
χ_4					
χ_5					

To obtain the last two characters, since we know that the permutation character is irreducible, we know that $\chi_{\text{sgn}}\cdot\chi_p$ is a new irreducible character by (1.5.3), and this character corresponds to the tensor product of their respective modules. So, we now have

S_4	1	(12)	(123)	(1234)	(12)(34)
χ_1	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
χ_p	3	1	0	-1	-1
$\chi_{\text{sgn}}\cdot\chi_p$	3	-1	0	1	-1
χ_5					

Since $\chi_5(1) = \dim_{\mathbb{C}} V$ (since this is the trace of the identity matrix), we can use the fact that $|G|$ equals the sum of the squares of the degrees of the irreducible characters to obtain $\chi_5(1)$. We have $24 = 1 + 1 + 3^2 + 3^2 + (\deg\,\chi_5)^2$, so $\deg\,\chi_5 = 2$. Now we can obtain the remaining values of the irreducible characters via the orthogonality of the columns:

S_4	1	(12)	(123)	(1234)	(12)(34)
χ_1	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
χ_p	3	1	0	-1	-1
$\chi_{\text{sgn}} \cdot \chi_p$	3	-1	0	1	-1
χ_5	2	0	-1	0	2

and we now have the complete character table of S_4.

The character table of S_5 is perhaps the first nontrivial one, since we will need to reach deeper into our toolbag of tricks to compute all the characters. The first few steps to create the character table of S_5 are the same as that of S_4; we first compute all the conjugacy classes and fill in the trivial, sign, and permutation characters, and then fill in the character of $\chi_p \otimes \chi_{\text{sgn}}$ as before:

S_5	1	(12)	(12)(34)	(123)	(123)(45)	(12345)	(1234)
χ_1	1	1	1	1	1	1	1
χ_{sgn}	1	-1	1	1	-1	1	-1
χ_p	4	2	0	1	-1	-1	0
$\chi_{\text{sgn}} \cdot \chi_p$	4	-2	0	1	1	-1	0
χ_5							
χ_6							
χ_7							

However, unlike in the character table of S_4, we now need to find a way to come up with an entirely new irreducible character χ_5. Note that once we do so, we can just multiply it with the sign character as usual to obtain χ_6, and then use the dimension formula + orthogonality to obtain χ_7.

For $1 \leq i \leq 5, 1 \leq j, \leq 5$, consider the set of unordered pairs

$$X = \{(i,j) \mid i \neq j\}.$$

Observe that S_5 acts on X in the natural way. Additionally, since $i, j \in \{1,2,3,4,5\}$, counting elements of X we see that $|X| = \frac{5 \cdot 4}{2} = 10$, so we've discovered an action of S_5 on a set of size 10. Thus if we take V to be the free $\mathbb{C}$-vector space with basis $\{v_x\}_{x \in X}$ indexed by our set of pairs X and linearly extend the action of S_5, we have a new $\mathbb{C}S_5$-module V. This is also a permutation module, though not the natural one, as our basis is now labeled by these pairs. Then with respect to our basis $\{v_x\}$ we obtain a matrix representation of S_5 in the same way as in our natural permutation module. In particular, since this is another permutation module, one can see that the character χ will be the number of fixed points again, but this time since S_5 is acting on X (which labels our basis of V), the character is the number of fixed points in X. The transposition $(12) \in S_5$ fixes the pairs $(3,4), (4,5), (1,5), (3,5)$, so there are 4 fixed points. The cycle (123) fixes $(4,5)$, the element (12)(34) fixes the pairs $(1,2), (3,4)$, and the element (12)(345) fixes $(1,2)$. The identity element fixes all 10 pairs. Putting these results into a table we have

S_5	1	(12)	(12)(34)	(123)	(123)(45)	(12345)	(1234)
χ	10	4	2	1	1	0	0

so we've found a new character of degree 10. The module V we created to obtain this character is sometimes called the symmetric part of $\chi_p^2 = \chi_p \otimes \chi_p$. The sizes of the centralizers of each representative (in the order as shown in the table) are 120, 12, 8, 6, 6, 5, and 4. Now using the inner product on characters we see that

$$\langle \chi, \chi \rangle = \frac{100}{120} + \frac{16}{12} + \frac{4}{8} + \frac{1}{6} + \frac{1}{6} = 3$$

and therefore by (1.3.4), since $3 = 1^2 + 1^2 + 1^2$, we know that $\chi = \psi_1 + \psi_2 + \psi_3$ where ψ_1, ψ_2, ψ_3 are three other irreducible characters of S_5. If we can compute three irreducible characters and find that we already know two of them, we can solve for the third. To find out what these are, we compute the inner product of χ with the other irreducible characters we've found from our table:

$$\langle \chi, \chi_1 \rangle = \frac{10}{120} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} = 1$$

$$\langle \chi, \chi_{\text{sgn}} \rangle = \frac{10}{120} - \frac{4}{12} + \frac{2}{8} + \frac{1}{6} - \frac{1}{6} = 0$$

$$\langle \chi, \chi_p \rangle = \frac{40}{120} + \frac{8}{12} + \frac{1}{6} - \frac{1}{6} = 1$$

$$\langle \chi, (\chi_{sgn} \cdot \chi_p) \rangle = \frac{40}{120} - \frac{8}{12} + \frac{1}{6} + \frac{1}{6} = 0$$

Hence by (1.3.4) $\chi = \chi_1 + \chi_p + \chi_5$ where χ_5 is some other irreducible character. Subtracting the values of χ_1 and χ_p from χ yields the values of χ_5 :

S_5	1	(12)	(12)(34)	(123)	(123)(45)	(12345)	(1234)
χ_5	5	1	1	-1	1	0	-1

so we've finally found a new irreducible character. At this point our character table of S_5 looks like:

S_5	1	(12)	(12)(34)	(123)	(123)(45)	(12345)	(1234)
χ_1	1	1	1	1	1	1	1
χ_{sgn}	1	-1	1	1	-1	1	-1
χ_p	4	2	0	1	-1	-1	0
$\chi_{\text{sgn}} \cdot \chi_p$	4	-2	0	1	1	-1	0
χ_5	5	1	1	-1	1	0	-1
χ_6							
χ_7							

Lastly, just like we did with the character table for S_4, we can multiply by the sign character, use the dimension formula to get the degree of χ_7, and then use column orthogonality to quickly fill in the final two rows:

S_5	1	(12)	(12)(34)	(123)	(123)(45)	(12345)	(1234)
χ_1	1	1	1	1	1	1	1
χ_{sgn}	1	-1	1	1	-1	1	-1
χ_p	4	2	0	1	-1	-1	0
$\chi_{\text{sgn}} \cdot \chi_p$	4	-2	0	1	1	-1	0
χ_5	5	1	1	-1	1	0	-1
χ_6	5	-1	1	-1	-1	0	1
χ_7	6	0	-2	0	0	1	0

and we're done.

At this point we've given a brief review of character theory and given some small n examples of the character table of S_n up to $n = 5$. For the character table of S_5, we had to find a completely new module χ_5 by considering an action on pairs (or, the symmetric part of the square of the natural permutation module). In general permutation modules are an excellent source of new characters to help fill out the character tables for higher n, however, overall beyond $n = 5$, filling out the character table in this way becomes highly nontrivial. The next section focuses on the theory of symmetric functions which is (for now) completely separate from representation/character theory, but it will turn out that the theory of symmetric functions can be used to yield the character table of S_n in general!

2 Symmetric Functions

2.1 Partitions

Definition 2.1.1: A *partition* λ is an infinite, weakly decreasing sequence

$$(\lambda_1, \lambda_2, \lambda_3, \dots), \lambda_i \in \mathbb{Z}_{\geq 0}, \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$$

with finitely many nonzero terms.

The nonzero λ_i are the called the *parts* of λ, and we call the number of nonzero parts the *length* of λ, denoted by $l(\lambda)$. We usually use the index n when referring to the length of a partition. The sum of the parts $d = \sum_i \lambda_i$ is denoted $|\lambda|$ and is called the *size* of λ. Usually the partitions are short enough so that we may just say that λ is a partition of size d and the length of λ is easily observed. We will also often write $\lambda \vdash d$, and in this case we will often abbreviate our speech and say "λ is a partition of d."

Example: The partition $\alpha = (4, 2, 2, 1, 0, 0, \dots)$ has 4 nonzero parts, so $l(\alpha) = 4$. The sum of the parts is $|\alpha| = 9$, so we write $\alpha \vdash 9$ and say α is a partition of 9.

Definition 2.1.2: An *n-part partition* λ is a weakly decreasing sequence $(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n)$ with finitely many terms.

Example: The sequence (2,1,0) is a 3-part partition of length 2.

Definition 2.1.3: Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition. The *truncation* of λ is denoted $\lambda^{[n]}$ and is defined as the n-part partition $\lambda^{[n]} = (\lambda_1, \lambda_2, \dots, \lambda_n)$ where λ_n is the last nonzero term in the sequence.

Example: $\lambda = (4, 2, 2, 1)$ is a 4 part partition of size 9. We will specify when λ is an n-part partition or a regular partition, but in both cases the notation will be the same since λ "knows" which one it is. However, we will always specify when we pass to a truncation, and the notation is given above. Later on this will arise when passing between an infinite variable ring of polynomials and a finite one, and we will always need to make clear which setting we are in. Following our example above, if $\alpha = (4, 2, 2, 1, 0, 0, \dots)$, then $\alpha^{[4]} = (4, 2, 2, 1)$, which is just our initial λ.

Definition 2.1.4: A *composition* μ is an infinite sequence $\mu = (\mu_1, \mu_2, \mu_3, \dots)$, $\mu_i \in \mathbb{Z}_{\geq 0}$ with finitely many nonzero terms.

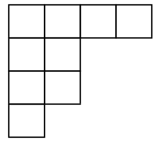
Just like with our definition for a partition λ, when we have a finite sequence $(\mu_1, \mu_2, \dots, \mu_n)$ of n elements as oppose to an infinite one, we say μ is an n-part composition.

Note that an n-part composition can have zeros in arbitrary places, so for compositions there is no distinction between "parts" and elements of the sequence as was defined for partitions. We are just saying that μ is finite and has n terms, some of which could be 0. Furthermore, note that in the case of a composition μ, the order of the μ_i does not matter, in contrast to a partition. For example, $(4, 5, 5, 6)$ is a 4-part composition, but it is not a partition, as the integers are not listed in weakly decreasing order.

The following is a nice way to visualize partitions:

Definition 2.1.5: Let λ be a partition. The *Young diagram* of λ is an array of boxes, with λ_i boxes in the *ith* row.

Example: If $\lambda = (4, 2, 2, 1)$ as before, the Young diagram for λ would be:



Definition 2.1.6: Given some partition λ of size d and length n, a *Young tableau* T (also called a Young tableau of shape λ, or a λ-tableau) is an array of boxes with n rows with the numbers $1, 2, \dots d$ filled into the boxes bijectively.

Example: If $\lambda \vdash 6 = (3, 2, 1)$ as before, then one such $\lambda-$tableau T would be:

1	5	4
2	6	
3		

Of course, if $d = |\lambda|$, there are $d!$ possible Young tableax of shape λ. If we allow for there to be repeats when we insert the numbers $1, 2, \dots, d$ into the boxes, then this would be called a *generalized* λ-tableau.

Example: Following the previous example again, one generalized λ-tableau would be:

1	5	5
1	3	
5		

Note that in this case we obtain two 1's, zero 2's, one 3, zero 4's, and two 5's - this gives rise to a 6-part composition $\mu = (2, 0, 1, 0, 3, 0)$. This μ is called the content of T, as given in the following definition:

Definition 2.1.7 : For T a generalized $\lambda-$tableau, we say the *content* of T is the n-part composition

$$\text{cont}(T) = (\mu_1, \mu_2, \dots, \mu_n)$$

where μ_i is the number of i's in T. Note here that n is the largest integer appearing in the tableau for λ, not necessarily equal to $|\lambda| = d$. In this case we say T has content μ. A generalized $\lambda-$tableau T with content μ is *semi-standard* if it is weakly increasing along its rows and strictly increasing down its columns.

Example: If $\lambda = (4, 2, 2) \vdash 8$ and $\mu = (0, 3, 0, 0, 1, 1, 3)$, then a semi-standard tableau of shape λ and content μ would be:

2	2	2	7
5	6		
7	7		

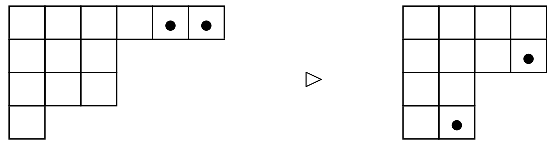
Before we can begin the discussion on symmetric functions, we need one more important definition involving partitions:

Definition 2.1.8 (Dominance Order): Let ν and λ be n-part partitions of size d. We write $\nu \lhd \lambda$ when we have:

$$\nu_1 + \nu_2 + \cdots + \nu_k \leq \lambda_1 + \lambda_2 + \dots \lambda_k$$

for all $1 \leq k \leq n$.

Example: Consider $\lambda = (6, 3, 3, 1)$, $\nu = (4, 4, 2, 2)$. These partitions have the following Young diagrams:



Where the dots indicate boxes which get moved down in the diagram on the right. This process of "moving down boxes" is what makes the diagram on the left larger than that on the right in this ordering. We see that the first row of the diagram of λ has more boxes than the first row for ν's diagram. The number of boxes in *both* the first and second row of λ is also greater than the number of boxes in the first two rows of ν. Continuing this process and verifying this total for each new row of λ is still larger than that of ν is exactly verifying the definition of our dominance order. Thus in this case we have $\nu \lhd \lambda$ and we say that λ dominates ν.

2.2 The Algebra of Symmetric Functions, Λ

Let $\{x_1, x_2, \ldots, x_n\}$ be a set of n variables. Let $\mathbb{C}[x_1, x_2, \ldots, x_n]$ be the algebra of polynomials in these variables. The symmetric group S_n acts on $\mathbb{C}[x_1, x_2, \ldots, x_n]$ by algebra homomorphisms given by permuting the variables, and the set of polynomials invariant under this action form a subalgebra, as given in the following definition:

Definition 2.2.1: Define $\Lambda_n \subset \mathbb{C}[x_1, x_2, \ldots, x_n]$ to be the subalgebra invariant under the action of $G = S_n$:

$$\Lambda_n = \mathbb{C}[x_1, x_2, \ldots, x_n]^{S_n}.$$

Lemma 2.2.2: Let Λ_n^d be the set of polynomials in Λ_n which are homogenous of degree d (homogeneous means the sum of the degrees of each variable in each monomial is constant.) Then we have:

$$\Lambda_n = \bigoplus_{d \geq 0} \Lambda_n^d$$

Proof. We have

$$\mathbb{C}[x_1, x_2, \ldots, x_n] = \bigoplus_{d \geq 0} \mathbb{C}[x_1, x_2, \ldots, x_n]^d.$$

That is, the polynomial algebra $\mathbb{C}[x_1, x_2, \ldots, x_n]$ is graded with each summand being the homogeneous degree d space. Since $G = S_n$ acts linearly on $\mathbb{C}[x_1, x_2, \ldots, x_n]$, the result follows.

$\square$

For an n-part composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \alpha_i \in \mathbb{Z}_{\geq 0}$ we denote x^α by the monomial

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

We now define our first species of symmetric polynomial.

Definition 2.2.3: Fix an n-part partition $\lambda = (\lambda_1, \ldots, \lambda_n)$. Define the symmetric polynomials $\{m_\lambda\}$ by:

$$m_\lambda(x_1, x_2, \ldots, x_n) = \sum_{\mu \sim \lambda} x^\mu.$$

Where the sum runs over all n-part compositions μ in the same S_n orbit as λ (S_n acts by permuting parts). In other words, m_λ is the "orbit sum" of the x^μ. For example: if $\lambda = (2,1,0)$ and $n = 3$, then $m_\lambda = x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2$.

Lemma 2.2.4: The $\{m_\lambda\}$ for all n-part partitions λ are a basis for Λ_n.

Proof. First note that $\mathbb{C}[x_1, \ldots, x_n]$ has a basis of monomials $x^\mu = x^{\mu_1} \cdots x_n^{\mu_n}$ for all n-part compositions μ. Now take any $f \in \mathbb{C}[x_1, \ldots, x_n]^{S_n} = \Lambda_n$. It is a linear combination of the $\{x^\mu\}$ as previously mentioned, and in particular since f is symmetric, the coefficient of x^μ must be equal to the coefficient of x^λ for $\lambda \sim \mu$. Hence f is a linear combination of $\{m_\lambda\}$ as well, which shows the $\{m_\lambda\}$ span. To see linear independence simply note that $m_\lambda = x^\lambda + \sum x^\mu$ where the sum runs over all the other μ such that $\mu \sim \lambda$. Since the $\{x^\lambda\}$ are linearly independent, so are the $\{m_\lambda\}$, which shows the $\{m_\lambda\}$ span and are also linearly independent, and therefore they are a basis for Λ_n.

□

It follows from Lemma 2.2.4 that $\dim \Lambda_n^d$ is the number of n-part partitions of d. We now define Λ, the algebra of symmetric functions in infinitely many variables. We wish to make this idea of freely passing to infinite variables precise, so we need some construction.

Consider the projection homomorphism $P_n : \Lambda_{n+1}^d \twoheadrightarrow \Lambda_n^d$ defined by evaluating at $x_{n+1} = 0$. It can be seen from the definition of m_λ that under this map P_n we have that m_λ is fixed if $l(\lambda) \leq n$ and $m_\lambda \mapsto 0$ otherwise. We now define Λ^d via an inverse limit of vector spaces relative to the homomorphisms P_n:

$$\Lambda^d = \varprojlim_n \Lambda_n^d$$

where elements of Λ^d are sequences $(f_n)_{n\geq 1} = (f_1, f_2, f_3, \ldots)$ where $f_n \in \Lambda_n^d$ and $P_n(f_{n+1}) = f_n$ for all n. In other words, to obtain an element $f \in \Lambda^d$, we need to find elements $f_n \in \Lambda_n^d$ which have the property that $P_n(f_{n+1}) = f_n$. An example of us going through this process of finding such elements for a specific type of symmetric polynomial will be seen in Section 1.3. Now set

$$\Lambda = \bigoplus_{d\geq 0} \Lambda^d$$

and we now have $\Lambda = \varprojlim \Lambda_n$ as an inverse limit of graded algebras, since $\Lambda_n = \bigoplus_{d\geq 0} \Lambda_n^d$ (Lemma 2.2.2). The graded algebra Λ is the algebra of symmetric functions countably infinite variables and we will need it to proceed, as it is often useful to work with infinitely many variables. To go along with our $\{m_\lambda\}$, we define a few more species of symmetric functions. When working in n variables:

Definition 2.2.5: For $r \geq 0$, the rth elementary symmetric function is

$$e_r = m_{(1^r)} = \sum_{1\leq i_1 < \cdots < i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r} \in \Lambda_n$$

Definition 2.2.6: For $r \geq 0$, the rth complete homogeneous symmetric functions is

$$h_r = \sum_{1\leq i_1 \leq \cdots i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r} \in \Lambda_n$$

.

Definition 2.2.7: For $r \geq 1$, the rth power sum symmetric function is

$$p_r = m_{(r)} = \sum_{i=1}^{n} x_i^r \in \Lambda_n$$

Generating functions for these symmetric functions are very useful and will be given soon. If we want to work in infinitely many variables, for example with elementary symmetric functions we can pass Λ by defining the corresponding symmetric functions $\{e_\lambda\}$ for each partition λ by:

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$$

and when λ is an n-part partition we write $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}$, and analogously for the other types of symmetric functions.

Theorem 2.2.8: The $\{e_\lambda\}$ for all partitions λ with $\lambda_1 \le n$ form a basis of Λ_n.

Proof. Multiplying out the product $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}$ we obtain a sum of monomials each of which is of the form:

$$(x_{i_1} x_{i_2} \cdots x_{i_n})(x_{j_1} x_{j_2} \cdots x_{j_n}) = x^\alpha$$

where $i_1 < i_2 < \cdots < i_{\lambda_1}, j_1 < j_2 < \cdots < j_{\lambda_2}$ and so on up through λ_n. It can be observed that in the Young diagram of λ, listing the numbers $i_1, i_2, \ldots i_{\lambda_1}$ down the first column and $j_1, j_2, \ldots j_{\lambda_2}$ down the second, and so on, for each $r \ge 1$ all of the symbols which are $\le r$ must occur in the top r rows. This shows that we have $\alpha_1 + \cdots + \alpha_r \le \lambda_1 + \ldots \lambda_r$ for each $r \ge 1$, so $\alpha \lhd \lambda$. Let λ^T be the partition obtained by reflecting the partition λ across its diagonal. Note that the monomial x^{λ^T} occurs exactly once by the above argument, and our proof of Lemma 2.2.4 shows that x^{λ^T} occurs exactly once in m_{λ^T}. This shows that m_{λ^T} occurs once in e_λ so that we have:

$$e_\lambda = m_{\lambda^T} + \sum_{\nu \lhd \lambda} a_{\lambda\nu} m_\nu$$

and since the $\{m_\lambda\}$ are a basis for Λ_n, the above expression for e_λ shows that the $\{e_\lambda\}$ are a basis as well. □

Note that from the above proof, we recover the fact that $e_{(r)} = m_{(1^r)}$ since as a partition, $(r)^T = (1^r)$.

Theorem 2.2.9: $\mathbb{C}[e_1, e_2, \ldots, e_n] \cong \Lambda_n$

Proof. Since the $\{m_\lambda\}$ are a basis for Λ_n and we have $e_\lambda = m_{\lambda^T} + \sum_{\nu \lhd \lambda} a_{\lambda\nu} m_\nu$, as before it follows that the $\{e_\lambda\}$ are also a basis for Λ_n. This shows that every element of Λ_n can be expressed as polynomial in the $\{e_n\}$. In other words, $\Lambda_n = \mathbb{C}[e_1, \ldots, e_n]$. □

It follows from Theorems 2.2.8 and 2.2.9 that when λ is an arbitrary partition, the $\{m_\lambda\}$ and $\{e_\lambda\}$ form a basis of Λ. Before we continue, we need to define an involution ω which switches h's and e's. We need some machinery for this. Recall that a generating function for some set of objects $\{a_r\}$ is a formal power series in a dummy variable t such that the coefficients of the series are exactly the $\{a_r\}$.

Lemma 2.2.10: For the rth complete symmetric functions $h_r = \sum_{|\lambda|=r} m_\lambda$ in infinitely many variables, the generating function is:

$$H(t) = \sum_{r \ge 0} h_r t^r = \prod_{i \ge 1} (1 - x_i t)^{-1}$$

Proof. Just note that $(1 - x_i t)^{-1} = \sum_{k \ge 0} x_i^k t^k$ and observe that when multiplying all of these series together, the coefficient on t^r will exactly be $\sum_{|\lambda|=r} m_\lambda$. □

Lemma 2.2.11: The generating function for the rth elementary symmetric function e_r in infinitely many variables is given by:

$$E(t) = \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t)$$

Proof. To see that the coefficient on t^r is e_r, observe that for each r, expanding out the product on the right gives:

$$(1 + x_1 t)(1 + x_2 t)(1 + x_3 t) \cdots (1 + x_r t) = 1 + (x_1 + x_2)t + \dots, + (x_{i_1} x_{i_2} \cdots x_{i_r})$$

since the sum runs over all $r \geq 0$, any coefficient of t_r will be obtained in this way for some sequence $i_1 < \cdots < i_r$, which exactly shows that the coefficient on t_r is $\sum\limits_{i_1 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} = e_r$.

□

Lemma 2.2.12:

$$H(t)E(-t) = 1 \quad \text{and} \quad \sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0.$$

Proof. $H(t)E(-t) = \prod\limits_{i \geq 1} (1 - x_i t)(1 - x_i t)^{-1} = 1$, which implies that $\sum\limits_{r \geq 0} (-1)^r e_r t^r \cdot \sum\limits_{r \geq 0} h_r t^r = 1$. In other words, the coefficient on every term besides the $r = 0$ term must be zero, hence $\sum\limits_{r=0}^{n} (-1)^r e_r h_{n-r} = 0$.

□

Note that by Theorem 2.2.9, the stability of the inverse limit implies that $\Lambda = \mathbb{C}[e_1, e_2, \dots]$. Due to this fact, it follows from Lemma 2.2.12 that $\Lambda = \mathbb{C}[h_1, h_2, \dots]$ as well.

Theorem 2.2.13: The map

$$\omega : \Lambda \longrightarrow \Lambda$$
$$e_r \longmapsto h_r$$

is an involution and an automorphism.

Proof. It is immediately an automorphism of algebras, as the $\{e_r\}$ and $\{h_r\}$ both generate Λ. To see that it is an involution, notice that Lemma 2.2.12 implies that $\omega(h_r) = e_r$, so $\omega(\omega(e_r)) = \omega(h_r) = e_r$, so $\omega^2 = \text{id}$.

□

The existence of the involution ω in Λ is one important advantage of Λ over Λ_n. The involution ω will come up frequently, and along with our generating functions, it is an essential tool. We will see in Section 3.4 that working in an infinite power series ring in two variables is invaluable, and the generating functions for our different families of symmetric functions become critical in order to prove one of the most important results about Λ.

2.3 Schur Polynomials and Schur Functions

As will be seen in Section 3, Schur Polynomials and Schur Functions play a key role in identifying the irreducible $\mathbb{C}S_n$-modules. In this section we construct the Schur polynomials $s_\lambda \in \Lambda_n$ and use the inverse limit to obtain the Schur functions $s_\lambda \in \Lambda$.

To begin, let us recall how we defined Λ_n. Suppose we have n variables $x_1, x_2, \ldots, x_n$. Then there is a natural action $S_n \circlearrowright \mathbb{C}[x_1, x_2, \ldots, x_n]$ whereby S_n acts on the polynomial algebra by algebra automorphisms permuting the variables:

$$g(x_i) = x_{g(i)},\ g \in S_n$$

and as defined before, the set of symmetric polynomials Λ_n are the polynomials which are fixed by this action. Our goal is to construct a new type of symmetric polynomial in the following way: Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be any n-part composition, corresponding to the monomial $x^\mu = x^{\mu_1} x^{\mu_2} \cdots x_n^{\mu_n}$. We now wish to obtain an anti-symmetric polynomial from x^μ called A_μ. This is done by defining A_μ:

$$A_\mu = \sum_{g \in S_n} \operatorname{sgn}(g) \cdot g(x^\mu)$$

where $g(x^\mu)$ is the action of g as defined above, and sgn is the sign (± 1) of g (1 when g is composed of an even number of transpositions and -1 when composed of an odd number of them).

Definition 2.3.1 A polynomial $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ is *anti-symmetric* for all $\sigma \in S_n$ if we have $\sigma \cdot f = \operatorname{sgn}(\sigma) f$ where S_n acts on the polynomial algebra $\mathbb{C}[x_1, x_2, \ldots, x_n]$ in the usual way.

Lemma 2.3.2: A_μ is anti-symmetric.

Proof. Let $\omega \in S_n$. Then for every $g \in S_n$ we have

$$\omega(A_\mu) = \sum_{g \in S_n} \omega[\sigma(g) \cdot g(x^\mu)] = \sum_{g \in S_n} \sigma(g) \cdot (\omega \cdot g)(x^\mu).$$

We see that if ω is an even permutation then $\omega \cdot g$ is even when g is even, and $\omega \cdot g$ is odd when g is odd. Thus when ω is even we have $\omega(A_\mu) = A_\mu$ Similarly if ω is odd then $\omega \cdot g$ is odd when g is even, and $\omega \cdot g$ is even when g is odd, so in this case when ω is odd we have $\omega(A_\mu) = -A_\mu$. Hence $\omega(A_\mu) = \operatorname{sgn}(\omega) A_\mu$. $\square$

Note that $A_\mu = 0$ unless $\mu_1, \ldots, \mu_n$ are all distinct, since the instant two elements u_i and u_j aren't distinct the transposition (i, j) will collapse the sum. Thus it is without loss of generality to take $\mu_1 > \mu_2 > \cdots > \mu_n \geq 0$ and write $\mu = \lambda + \rho$ where λ is an arbitrary partition of at most n parts and $\rho = (n-1, n-2, \ldots, 1, 0)$. We can now write:

$$A_\mu = A_{\lambda+\rho} = \sum_{g \in S_n} \operatorname{sgn}(g) \cdot g(x^{\lambda+\rho})$$

In other words, A_μ is the following determinant:

$$A_{\lambda+\rho} = \det(x_i^{\lambda_j - n + j}) = \det(x_i^{\mu_j})$$

$$= \det \begin{bmatrix} x_1^{\mu_1} & x_1^{\mu_2} & \cdots & x_1^{\mu_n} \\ x_2^{\mu_1} & x_2^{\mu_2} & \cdots & x_2^{\mu_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\mu_1} & x_n^{\mu_2} & \cdots & x_n^{\mu_n} \end{bmatrix}$$

We recover the fact that $A_\mu = 0$ when any of the parts of μ are indistinct as this would result is the above matrix having two equivalent rows, giving a zero determinant. When we consider A_ρ we obtain:

$$A_\rho = \det(x_i^{n-j})$$

$$= \det \begin{bmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & 1 \end{bmatrix}$$

This is now recognized to be the Vandermonde determinant:

$$A_\rho = \prod_{1 \le i < j \le l} (x_i - x_j)$$

It's clear from this product form of A_ρ that it is another anti-symmetric polynomial. Before we can define the Schur polynomials and the Schur functions $\{s_\lambda\}$ we need the following fact:

Lemma 2.3.3: $A_{\lambda+\rho}$ is divisible by A_ρ as polynomials in $\mathbb{C}[x_1, x_2, \ldots, x_n]$. Furthermore, the quotient $\frac{A_{\lambda+\rho}}{A_\rho}$ is symmetric, i.e, $\frac{A_{\lambda+\rho}}{A_\rho} \in \Lambda_n$.

Proof. As previously noted, plugging in x_i for any $x_j (j \neq i)$ in $A_{\lambda+\rho}$ results in $A_{\lambda+\rho} = 0$. This means that $A_{\lambda+\rho}$ contains factors $(x_i - x_j)$ for all $i \neq j$, and hence is divisible by their product which is exactly A_ρ. Then since both $A_{\lambda+\rho}$ and A_ρ are anti-symmetric their quotient must be symmetric.

□

Definition 2.3.4: Given a partition λ of length at most n, the corresponding *Schur polynomial* in n variables s_λ is defined as:

$$s_\lambda \equiv \frac{A_{\lambda+\rho}}{A_\rho}$$

To be clear, λ is a regular (infinite) partition with at most n parts (nonzero terms). We will now denote the s_λ by $s_\lambda^{[n]}$ so as to specify when we are in the n variable setting. As before, by $s_\lambda^{[n]}$ we mean the n-part partition obtained by truncating all of the zeros, that is, keeping only the first n parts of λ. In order to properly define the Schur *function* as opposed to the Schur polynomial in n variables we need to check that what we will define as s_λ is stable in Λ, which has infinitely many variables. First note the following:

Lemma 2.3.5: Recall the projection map P_n defined on page 15; It is an algebra homomorphism given by equating the last variable x_{n+1} to zero. Under this map P_n we have $s_\lambda^{[n+1]} \mapsto s_\lambda^{[n]}$ when the $(n+1)th$ term of $\lambda^{[n+1]}$ is zero, and $s_\lambda^{[n+1]} \mapsto 0$ otherwise.

Proof. It is clear that if the $(n+1)th$ term of $\lambda^{[n+1]}$ is nonzero, then $P_n(s_\lambda^{[n+1]}) = 0$ since every monomial in $s_\lambda^{[n+1]}$ has an x_{n+1} in it. Otherwise, observe that if $x_{n+1} \mapsto 0$, the determinant in $A_{\lambda+\rho}$ takes the following form:

$$= \det \begin{bmatrix} x_1^{\mu_1} & x_1^{\mu_2} & \cdots & 1 \\ x_2^{\mu_1} & x_2^{\mu_2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Where as before $\mu = \lambda + \rho$. The last column is all 1's now because $\mu_{n+1} = 0$. Now expanding along the bottom row to compute the determinant of this matrix just results in $1 \cdot \det(x_i^{\lambda_j - n + j})$, where this λ is now an n part partition as oppose to an $n+1$ part partition. This means that $A^{[n+1]}_{\lambda+\rho} \mapsto A^{[n]}_{\lambda+\rho}$. Since the final entry of ρ is 0, ρ is fixed, and the result is:

$$P_n(s_\lambda^{[n+1]}) = \frac{A^{[n+1]}_{\lambda+\rho}}{A^{[n+1]}_{\rho}} = \frac{\det(x_i^{\lambda_j - n + j})}{A_\rho} = \frac{A^{[n]}_{\lambda+\rho}}{A^{[n]}_{\rho}} = s_\lambda^{[n]}$$

□

Since $s_\lambda^{[n]} \in \Lambda_n$ is stable in Λ, we define the Schur function corresponding to λ (which has infinitely many variables) in the following way:

Definition 2.3.6: For λ partition of length at most n, the associated *Schur function* is:

$$s_\lambda \equiv \varprojlim s_\lambda^{[n]}$$

where $\varprojlim$ is the inverse limit as defined in Section 2.2, i.e., to obtain $s_\lambda \in \Lambda$ it is well defined to allow the partition $\lambda^{[n]}$ to pass to the infinite partition λ by adding an infinite string of zeros on the end.

The Schur functions give another basis for Λ (proven in 2.4) and they are central to the rest of this document. Before we continue, we need a few more results. Let us once again work in n variables $x_1, \ldots, x_n$. Let e_r^k denote the elementary symmetric functions in the variables $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$ and let M be the $n \times n$ matrix with ij-entry $(-1)^{n-j} e^j_{n-i}$.

Lemma 2.3.7: Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be an n-part composition. Define the matrices $A_\alpha = (x_j^{\alpha_i})$ for $i, j = 1, \ldots, n$, and $H_\alpha = (h_{\alpha_i - n + j})$ for $i, j = 1, \ldots, n$. Then $A_\alpha = H_\alpha \cdot M$.

Proof. Using our generating function $E(t)$, define

$$E^k(t) = \sum_{r=0}^{n-1} e_r^k t^r = \prod_{i \neq k} (1 + x_i t).$$

Now using our generating function $H(t)$ we have

$$H(t)E^k(-t) = \prod_{i \geq 1} \prod_{i \neq k} (1 - x_i t)(1 - x_i t)^{-1}.$$

Note that this product equals 1 on every term except when $i = k$, which shows $H(t)E^k(-t) = (1 - x_k t)^{-1}$. Now applying a formal Taylor series:

$$H(t)E^k(-t) = \sum_{r \geq 0} h_r t^r \cdot \sum_{r=0}^{n-1} e_r^k t^r = 1 + x_k t + x_k^2 t^2 + \ldots$$

the above shows that the coefficient on t^{α_i} on the right hand side will be x^{α_i} and hence:

$$\sum_{j=1}^{n} h_{\alpha_i - n + j} \cdot (-1)^{n-j} e^k_{n-j} = x_k^{\alpha_i}$$

thus the ij entry in the matrix A_α is exactly the ij entry in $H_\alpha \cdot M$, so $H_\alpha \cdot M = A_\alpha$.

□

Theorem 2.3.8: $a_\alpha = \det(A_\alpha) = A_\rho \sum\limits_{\sigma \in S_n} \operatorname{sgn}(\sigma) h_{\alpha - \sigma(\rho)}$.

Proof. By Lemma 2.3.7, if we take determinants we have $\det(A_\alpha = \det(H_\alpha) \cdot \det(M)$. Notice that $\det(M)$ must equal A_ρ since when $\lambda = (0, 0, \dots)$ we have $\det(H_\alpha) = \det(H_\rho) = 1$. Thus $a_\alpha = A_\rho \cdot \det(H_\alpha)$, so we have

$$a_\alpha = A_\rho \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) h_{\alpha - \sigma(\rho)}$$

. □

Corollary 2.3.9: We have $s_\lambda = \sum\limits_{\sigma \in S_n} \operatorname{sgn}(\sigma) h_{\lambda + \rho - \sigma(\rho)}$.

Proof. By Theorem 2.3.8, if we let $\alpha = \lambda + \rho$ and divide by A_ρ we obtain

$$\frac{A_{\lambda+\rho}}{A_\rho} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) h_{\lambda + \rho - \sigma(\rho)}$$

and the left hand side is the definition of s_λ.

□

The above formula can be realized as a determinant of a matrix:

$$s_\lambda = \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+n-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_n-n+1} & h_{\lambda_n-n+2} & \cdots & h_{\lambda_n} \end{bmatrix}.$$

2.4 The Complex Inner Product on Λ

We now define a sesquilinear form on Λ (i.e., linear in the first component, antilinear in the second) by having the bases $\{h_\lambda\}$ and $\{m_\lambda\}$ be dual to eachother:

Definition 2.4.1: The inner product $\langle \cdot, \cdot \rangle$ on Λ is defined by:

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$$

for partitions λ, μ where $\delta_{\lambda\mu}$ is the Kronecker delta.

Before we prove orthogonality relations, we need some slightly technical results about generating functions and a few lemmas. We closely follow Sections 1.2-1.4 of MacDonald [**Mac79**] for all of these results. As is the goal of this document, we list only the necessary results from those Chapters so that we may quickly bring our focus to the Kostka matrix.

Lemma 2.4.2: Recall that for each $r \geq 1$, the rth power sum is defined as $p_r = \sum\limits_i x_i^r = m_{(r)}$. The generating function for the $\{p_r\}$ is:

$$P(t) = \sum_{i \geq 1} \frac{d}{dt} \log \frac{1}{1 - x_i t}$$

and furthermore:

$$P(t) = H'(t)/H(t)$$

Proof. By definition of $P(t)$ and p_r we have

$$P(t) = \sum_{r\geq 1} p_r t^{r-1} = \sum_{i\geq 1}\sum_{r\geq 1} x_i^r t^{r-1}.$$

Once again using a formal Taylor series:

$$P(t) = \sum_{i\geq 1} \frac{x_i}{1-x_i t}$$

Now just using log rules we obtain

$$P(t) = \frac{d}{dt}\log\prod_{i\geq 1}(1-x_i t)^{-1} = \frac{d}{dt}\log H(t) = H'(t)/H(t).$$

□

Recall our formula z_λ for $\lambda \vdash n$ and g an element of cycle type λ in S_n from Section 1:

$$z_\lambda = \prod_{i\geq 1} i^{m_i}\cdot m_i!$$

we showed in Section 1 that $z_\lambda = |C(g_\lambda)|$, the size of the centralizer of an element of cycle type λ in S_n. This number is actually very crucial, and will come up a lot. In the realm of symmetric functions, we first encounter it here:

Lemma 2.4.3:

$$H(t) = \sum_\lambda z_\lambda^{-1} p_\lambda t^{|\lambda|}$$

Proof. From Lemma 2.4.2 we have $\frac{d}{dt}\log H(t) = P(t)$. Formally integrating $P(t) = \sum_{r\geq 1} p_r t^{r-1}$ with respect to t we obtain $\sum_{r\geq 1}\frac{p_r t^r}{r} = \log H(t) \Rightarrow H(t) = \exp\sum_{r\geq 1} p_r t^r/r$. By exponent rules this is just $\prod_{r\geq 1}\exp(p_r t^r/r)$. Then a formal Taylor series yields:

$$\prod_{r\geq 1}\sum_{m_r=0}^{\infty}(p_r t^r)^{m_r}/(r^{m_r}\cdot m_r!)$$

which by observation is just $\sum_\lambda z_\lambda^{-1} p_\lambda t^{|\lambda|}$.

□

Corollary 2.4.4: In the finite variable case we have

$$h_n = \sum_{\lambda\vdash n} z_\lambda^{-1} p_\lambda.$$

Proof. This follows immediately from Lemma 2.4.3.

□

Lemma 2.4.5: Consider the expression $\prod_{i,j}(1-x_iy_j)^{-1}$ in the set of variables $\{x_i\},\{y_j\}$. We have:

$$(1) \qquad \prod_{i,j}(1-x_iy_j)^{-1} = \sum_{\lambda} z_{\lambda}^{-1}p_{\lambda}(x)p_{\lambda}(y)$$

$$(2) \qquad \prod_{i,j}(1-x_iy_j)^{-1} = \sum_{\lambda} h_{\lambda}(x)m_{\lambda}(y) = \sum_{\lambda} m_{\lambda}(x)h_{\lambda}(y)$$

$$(3) \qquad \prod_{i,j}(1-x_iy_j)^{-1} = \sum_{\lambda} s_{\lambda}(x)s_{\lambda}(y)$$

Proof. To prove (1), we recall that $H(t) = \prod_{i\geq 1}(1-x_it)^{-1}$, so simply applying Lemma 2.4.3 to the variables $\{y_j\}$ gives the result. For (2), we have

$$\prod_{i\geq 1}(1-x_iy_j)^{-1} = \prod_j H(y_j) = \prod_j \sum_{r=0}^{\infty} h_r(x)y_j^r = \sum_{\alpha} h_{\alpha}(x)y^{\alpha} = \sum_{\lambda} h_{\lambda}(x)m_{\lambda}(y)$$

and following the same reasoning switching the variables x_i and y_j proves the second equality in (2). To prove equation (3), work in two finite sets of variables $\{x_1,\ldots,x_n\}$ and $\{y_1,\ldots,y_n\}$. As in Section 3, let $\rho = (n-1,n-2,\ldots,0)$, and $a_{\rho}(x)$, $a_{\rho}(y)$ be as in Theorem 2.3.8. Then by equation (2) we have

$$\begin{aligned}
a_{\rho}(x)a_{\rho}(y)\prod_{i,j=1}^{n}(1-x_iy_j)^{-1} &= a_{\rho}(x)\cdot a_{\rho}(y)\cdot \sum_{\lambda\vdash n} h_{\lambda}(x)m_{\lambda}(y) \\
&= a_{\rho}(x)\cdot \sum_{\mu}\sum_{\sigma\in S_n} h_{\mu}(x)\mathrm{sgn}(\sigma)\cdot y^{\mu+\sigma(\rho)} \\
&= a_{\rho}(x)\cdot \sum_{\beta}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)h_{\beta-\sigma(\rho)}(x)y^{\beta} \\
&= \sum_{\beta} a_{\beta}(x)y^{\beta}
\end{aligned}$$

where the sum in line 2 is over μ an n-part composition, and equality holds by Theorem 2.3.8. Now since $a_{\sigma(\beta)} = \mathrm{sgn}(\sigma)a_{\beta}$, we may re-index the sum to be over λ an n-part partiton to obtain:

$$\sum_{\beta} a_{\beta}(x)y^{\beta} = \sum_{\lambda} a_{\lambda+\rho}(x)a_{\lambda+\rho}(y) = a_{\rho}(x)a_{\rho}(y)\sum_{\lambda} s_{\lambda}(x)s_{\lambda}(y)$$

Hence $\prod_{i,j=1}^{n}(1-x_iy_j)^{-1} = \sum_{\lambda} s_{\lambda}(x)s_{\lambda}(y)$.

□

Lemma 2.4.6: Let $\{u_{\lambda}\},\{v_{\lambda}\}$ be bases of Λ_n^d indexed by partitions $\lambda\vdash d$ for each $d\geq 0$. Then the following are equivalent:

$$(1) \qquad \langle u_{\lambda}, v_{\lambda}\rangle = \delta_{\lambda\mu} \text{ for all } \lambda,\mu$$

$$(2) \qquad \sum_{\lambda} u_{\lambda}(x)v_{\lambda}(y) = \prod_{i,j}(1-x_iy_j)^{-1}$$

Proof. Since the $\{h_r\}$ and $\{m_\lambda\}$ are bases we may write $u_\lambda = \sum_\rho a_{\lambda\rho} h_\rho$ and $v_\mu = \sum_\sigma b_{\mu\sigma} m_\sigma$. By definition of the inner product we see that $\langle u_\lambda, v_\mu \rangle = \sum_\rho a_{\lambda\rho} b_{\mu\rho} \Rightarrow$ (1) is equivalent to $\sum_\rho a_{\lambda\rho} b_{\mu\rho} = \delta_{\lambda\mu}$. Now note that by the second equation from Lemma 2.4.5, we have that (2) is the same as $\sum_\lambda u_\lambda(x) v_\lambda(y) = \sum_\rho h_\rho(x) m_\rho(y)$, and by matching coefficients this is the same as $\sum_\lambda a_{\lambda\rho} b_{\lambda\sigma} = \delta_{\rho\sigma}$, which shows that (1) and (2) are equivalent.

□

Theorem 2.4.7: We have the following orthogonality relations:

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}, \langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$$

.

Proof. By Lemma 2.4.5 equation (1) we know that $\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y)$, so applying Lemma 2.4.6 shows that $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$. Similarly by Lemma 2.4.5 equation (3) we have $\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_\lambda s_\lambda(x) s_\lambda(y)$, so once again by Lemma 2.4.6 we have $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$.

□

Theorem 2.4.7 shows that the $\{s_\lambda\}$ are an orthonormal basis for Λ, and the $\{p_\lambda\}$ are an orthogonal basis for Λ.

2.5 The Kostka Matrix

The goal of this section is to prove Young's rule, which yields a transition matrix between the two bases $\{s_\lambda\}$ and $\{m_\lambda\}$. Once this matrix is obtained, as will be seen in a future chapter, it will allow us to compute the irreducible characters of S_n for any n and obtain its character table. Before we prove Young's rule we need a few results.

Lemma 2.5.1: (Pieri Formula)

$$s_\lambda \cdot e_r = \sum_\mu s_\mu$$

where the sum runs over all partitions μ which are obtained by adding r 1's to λ, i.e, adding a box to the end of r different rows in the Young diagram of λ.

Proof. By definition of s_λ we have $s_\lambda = \frac{A_{\lambda+\rho}}{A_\rho}$, so we want to show $\frac{A_{\lambda+\rho}}{A_\rho} \cdot e_r = \sum_\mu s_\mu$, which is the same as showing $A_{\lambda+\rho} \cdot e_r = \sum_\mu s_\mu A_\rho$. But once again using the definition of s_μ the sum just becomes $\sum_\mu A_{\mu+\rho}$. Thus we seek to show:

$$A_{\lambda+\rho} \cdot e_r = \sum_\mu A_{\mu+\rho}$$

By the alternative definition of determinant and rewriting $\{e_r\}$:

$$A_{\lambda+\rho} \cdot e_r = \sum_{g \in S_n} \operatorname{sgn}(g) x^{g(\lambda+\rho)} \cdot \sum_\alpha x^\alpha$$

where $\alpha_i \in \{0,1\}, |\alpha| = r, \alpha$ an n-part partition. Then:

$$\begin{aligned}
A_{\lambda+\rho} \cdot e_r &= \sum_{g \in S_n} \operatorname{sgn}(g) x^{g(\lambda+\rho)} \cdot \sum_{\alpha} x^{\alpha} \\
&= \sum_{g \in S_n} \sum_{\alpha} \operatorname{sgn}(g) x^{g(\lambda+\rho)} x^{g(\alpha)} \\
&= \sum_{\alpha} \sum_{g \in S_n} \operatorname{sgn}(g) x^{g(\lambda+\alpha+\rho)} \\
&= \sum_{\alpha} A_{\lambda+\alpha+\rho}
\end{aligned}$$

Where the second equality holds due to the fact that $\{x^{\alpha}\}$ equals $\{x^{g(\alpha)}\}$ as sets, and the last equality by definition of the determinant again. Now note that since α is comprised of 0's and 1's, by definition of the μ partitions we have that the $\lambda + \alpha$ partitions are exactly the μ's. Hence:

$$A_{\lambda+\rho} \cdot e_r = \sum_{\mu} A_{\mu+\rho}$$

and we are done.

$\square$

Corollary 2.5.2: We have

$$s_{\lambda} \cdot h_r = \sum_{\mu} s_{\mu}$$

where the sum runs over all partitions μ obtained from λ by adding a box to the bottom of r different columns.

Proof. Just apply the involution ω as defined in Theorem 2.2.13 to e_r in Lemma 2.5.1.

$\square$

Definition 2.5.3: Let λ be a partition and let $h_r^* : \Lambda \longrightarrow \Lambda$ be the linear map defined by:

$$h_r^*(s_{\lambda}) = \sum_{\mu} s_{\mu}$$

where the sum runs over all partitions μ obtained from λ by removing a box from the bottom of r different columns.

Lemma 2.5.4: The map h_r^* is adjoint to multiplication by h_r with respect to $\langle \cdot, \cdot \rangle$, i.e:

$$\langle f, h_r g \rangle = \langle h_r^*(f), g \rangle$$

Proof. It suffices to check this condition for $f = s_{\lambda}$, $\lambda \vdash d + r$ and $g = s_{\mu}$, $\mu \vdash d$, that is:

$$\langle h_r^*, s_{\mu} \rangle = \langle s_{\lambda}, h_r \cdot s_{\mu} \rangle$$

Note that by Corollary 2.5.2, the right hand side is just $\langle s_\lambda, \sum_\mu s_\mu \rangle$, where this sum runs over the μ's which add a box to the bottom of r different columns. By orthogonality this is entirely zero unless λ can be obtained from μ in this way, and in that case it is 1. The left hand side is $\langle \sum_\mu s_\mu, s_\mu \rangle$, but in this case the sum runs over μ's by removing boxes from λ. This means that the left hand side is again zero unless λ can be obtained from μ by adding a box to r different rows. Thus the left and right hand sides are the exact same.

□

Definition 2.5.5: Let λ be a partition and μ a composition. The *Kostka number* $K_{\lambda\mu}$ is the number of semistandard tableaux of shape λ and content μ.

Example: Let $\lambda = (3, 2) \vdash 5$ and let $\mu = (2, 2, 1)$. Then to count the number of semistandard Young tableau with shape $(3, 2)$ and content $(2, 2, 1)$, we must fill in the Young tableau corresponding to λ with two 1's, two 2's, and one 3

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$$

such that the rows are weakly increasing and the columns are strictly increasing. We must put a 1 in the top left, since anything else in that spot would lead to us violating strictly increasing columns, and then it follows that the other 1 must go to the right of the previous. Furthermore, we see that we must place a 2 below the top left 1 so as to follow the same rule. At this point we have

$$\begin{array}{|c|c|c|}\hline 1 & 1 & \\ \hline 2 & & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$$

and now its clear that the remaining 2 can go in either slot, and the same case for the 3. So there are two total semistandard young tableaux of shape $\lambda = (3, 2)$ and content $\mu = (2, 2, 1)$:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \qquad\qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$$

so in this case $K_{\lambda\mu} = 2$.

Theorem 2.5.6:

$$\langle s_\lambda, h_\mu \rangle = K_{\lambda\mu}.$$

Proof. First, by Lemma 2.5.4, we may write $\langle s_\lambda, h_\mu \rangle$ as $\langle s_\lambda, h_{\mu_1} h_{\mu_2} \cdots h_{\mu_l} \rangle = \langle h^*_{\mu_l} h^*_{\mu_{l-1}} \cdots h^*_{\mu_1}(s_\lambda), 1 \rangle$. Note that $1 = s_\emptyset$, and each application of h^*_r in the inner product removes a box from μ_i rows in λ. Then since $\sum_i \mu_i$ is the number of boxes in λ, this means that the process of applying h^*_r will eventually result in $k \cdot \langle s_\emptyset, s_\emptyset \rangle$, $k \in \mathbb{Z}_{\geq 0}$. By orthonormality of the Schur functions this is just k. It remains to show that $k = K_{\lambda\mu}$. We show this with the following process:

Applying $h^*_{\mu_l}$, we obtain some new number of λ diagrams with a box removed from the bottom of μ_l different columns. Replace this box with the number l. Continue this process for μ_2, μ_3, and so on, replacing the removed boxes with the numbers $l-1, l-2$, etc. It is clear that all of the resulting tableaux from this process will have weakly increasing rows. But in fact, the columns will also be strictly increasing. This is because of the definition of h^*_r: we require that the boxes be removed from distinct columns. Thus the tableaux which could be obtained by removing boxes in the same column are not counted, which means we will never have a stacking of the same number on top of eachother among our sum of tableaux. This exactly yields only tableaux which have strictly increasing columns. Hence our final integer k is precisely the number of semistandard λ-tableaux with content μ.

□

We now have what we need to prove Young's rule.

Theorem 2.5.7 (Young's Rule):

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$$

for a partition λ and a composition μ.

Proof. Since the m_μ are a basis for Λ, we may write $s_\lambda = \sum\limits_\nu a_{\lambda\nu} m_\nu$ for some unknown coefficients $a_{\lambda\nu}$ indexed by compositions ν. Computing the inner product with h_μ on the left and right hand sides gives:

$$\langle s_\lambda, h_\mu \rangle = \sum_\nu a_{\lambda\nu} \langle m_\nu, h_\mu \rangle$$

The left hand side is $K_{\lambda\mu}$ as given by Theorem 2.5.6, and by definition of $\langle \cdot, \cdot \rangle$ on Λ the right hand side is $\sum\limits_\nu a_{\lambda\nu} \delta_{\mu\nu}$ which is just $a_{\lambda\mu}$. Hence our coefficients $a_{\lambda\mu}$ are exactly $K_{\lambda\mu}$, which proves Young's Rule.

□

Corollary 2.5.8: If working in n variables we have:

$$s_\lambda = \sum_T x^{\text{cont}(T)} \in \Lambda_n$$

Where λ is an n part partition, the sum runs over all semistandard $\lambda-$tableaux T, and

$$x^{\text{cont}(T)} = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}$$

where μ_i is the number of $i's$ in T.

Proof. By Young's Rule we have $s_\lambda = \sum\limits_\mu K_{\lambda\mu} m_\mu$ for λ, μ both n-part partitions. Note that when in n variables we may write $m_\mu = \sum\limits_{\nu \sim \mu} x^\nu$. Using this re-indexes our sum:

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu = \sum_{\mu'} K_{\lambda\mu'} x^{\mu'}$$

The sum on the right hand side is now indexed by n-part compositions as a result. Furthermore our coefficients are $K_{\lambda\mu}$, which shows that we exactly have:

$$s_\lambda = \sum_T x^{\mathrm{cont}(T)}$$

$\square$

Young's Rule gives us a transition matrix $(K_{\lambda\mu})$ from the basis $\{m_\lambda\}$ to $\{s_\mu\}$, called the Kostka Matrix. It is indexed by partitions and is uni-triangular. The Kostka Matrix will be elaborated upon in Section 3 where it comes into play. For now, here are some examples of Kostka matrices for $n = 3, 4, 5, 6$:

$$\begin{array}{cc} & \begin{array}{ccc} (3) & (2,1) & (1,1,1) \end{array} \\ \begin{array}{c} (3) \\ (2,1) \\ (1,1,1) \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccccc} (4) & (3,1) & (2,2) & (2,1,1) & (1,1,1,1) \end{array} \\ \begin{array}{c} (4) \\ (3,1) \\ (2,2) \\ (2,1,1) \\ (1,1,1,1) \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccccccc} (5) & (4,1) & (3,2) & (3,1,1) & (2,2,1) & (2,1,1,1) & (1,1,1,1,1) \end{array} \\ \begin{array}{c} (5) \\ (4,1) \\ (3,2) \\ (3,1,1) \\ (2,2,1) \\ (2,1,1,1) \\ (1,1,1,1,1) \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccccccc} (6) & (5,1) & (4,2) & (4,1,1) & (3,3) & (3,2,1) & (3,1,1,1) & (2,2,2) & (2,2,1,1) & (2,1,1,1,1) & (1,1,1,1,1,1) \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 & 2 & 3 & 3 & 4 & 6 & 9 \\ 0 & 0 & 0 & 1 & 0 & 1 & 3 & 1 & 3 & 6 & 10 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 4 & 8 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 4 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

3 The Characteristic Map

3.1 The Space of Class Functions, $R(S_n)$

How do we proceed in using Λ to identify the irreducible characters of the symmetric group in general? This section is dedicated to constructing the "characteristic map", which will lead to a correspondence between irreducible characters of S_n and one of our bases for Λ.

Definition 3.1.1: Let $f : S_n \to \mathbb{C}$ be a function which is constant on the conjugacy classes of S_n. As in Section 1, we denote the set of all such functions for a fixed n by $R(S_n)$.

With addition and scalar multiplication defined pointwise, $R(S_n)$ is a vector space over $\mathbb{C}$. Observe that $\dim R(S_n) = p(n)$, as the number of conjugacy classes in the symmetric group is $p(n)$. Recall our indicator functions from Section 1 (page 5):

$$\delta_\mu(g_\lambda) = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{else} \end{cases}$$

where $g_\lambda \in S_n$ is an element of cycle type λ, and $\mu \vdash n$. It is clear that this is a basis for $R(S_n)$ from its definition. We also want to define an inner product on $R(S_n)$: this will be the standard inner product for characters (recall this from Definition 1.3.1).

Proposition 3.1.2: The indicator functions are orthogonal with respect to the above form, with

$$\langle \delta_\mu, \delta_\mu \rangle = \frac{1}{z_\mu}.$$

Proof. If we take $\lambda \neq \mu$, we have $\langle \delta_\mu, \delta_\lambda \rangle = \frac{1}{n!} \sum_{g \in S_n} \delta_\mu(g)\overline{\delta_\lambda(g)}$. But for each conjugacy class, one of δ_μ or δ_λ must be zero on that class since $\lambda \neq \mu$, so $\langle \delta_\mu, \delta_\lambda \rangle = 0$ in this case. If $\lambda = \mu$, then

$$\langle \delta_\mu, \delta_\mu \rangle = \frac{1}{n!} \sum_{g \in S_n} \delta_\mu(g)\overline{\delta_\mu(g)} = \frac{1}{n!} \sum_{g \in C_\mu} 1 = \frac{|C_\mu|}{n!} = \frac{1}{z_\mu}.$$

where the last equality holds by Orbit-Stabilizer theorem (with conjugation action) and by our proof of Lemma 1.4.3 that z_μ is the size of the centralizer of an element of cycle type μ (centralizer is the stabilizer in this case). Thus $\langle \delta_\mu, \delta_\mu \rangle = \frac{1}{z_\mu}$.

□

Definition 3.1.3: The *nth* characteristic map is defined as:

$$\mathrm{ch}_n : R(S_n) \to \Lambda^n$$
$$\chi \mapsto \sum_{\mu \vdash n} z_\mu^{-1} \chi(g_\mu) p_\mu$$

where g_μ is an element of cycle type μ and $\chi(g_\mu)$ is the evaluation of χ on g_μ, and z_μ is as defined for Proposition 1.4.3. We now have the following result:

Theorem 3.1.4: ch_n is an isometric isomorphism of vector spaces.

Proof. It's clear that ch_n is linear. Note that $\mathrm{ch}_n(\delta_\mu) = \frac{1}{z_\mu} p_\mu$, so ch_n sends a basis to a basis bijectively, and thus is an isomorphism of vector spaces. It suffices to check that ch_n is an isometry by checking the inner product on a basis. By Proposition 3.1.2 we have

$$\langle \delta_\lambda, \delta_\mu \rangle = \delta_{\lambda\mu} \frac{1}{z_\lambda} = \left\langle \frac{p_\lambda}{z_\lambda}, \frac{p_\mu}{z_\mu} \right\rangle = \langle \mathrm{ch}_n(\delta_\lambda), \mathrm{ch}_n(\delta_\mu) \rangle$$

since $\mathrm{ch}_n(\delta_\lambda) = \frac{p_\lambda}{z_\lambda}$ and $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$. Thus ch_n is both an isomorphism of vector spaces and an isometry.

□

3.2 The Characteristic Map

Let us define

$$R = \bigoplus_{n \geq 0} R(S_n).$$

The space $R(S_n)$ is an algebra with pointwise multiplication corresponding to the tensor product as in Section 1.5. We will define a new "external" product $\circ$ on R, so that R may be viewed as a graded algebra with respect to $\circ$ instead.

Note that if χ and ψ are characters of S_n and S_m respectively, then $\chi \boxtimes \psi$ is a character of $S_n \times S_m \leq S_{n+m}$, where $(\chi \boxtimes \psi)(g, h) = \chi(g)\psi(h)$. This motivates the next construction:

Definition 3.2.1 (Induction and Restriction): Let $H \leq S_n$ and let χ be a class function on S_n. We can form a class function on H via the map

$$\chi{\downarrow_H}(h) = \chi(h)$$

which is called the restriction of χ to $H \leq S_n$. Similarly, if ψ is instead a class function on H, we can form a class function on S_n via the map

$$\psi{\uparrow^{S_n}}(g) = \frac{1}{|H|} \sum_{x \in G} \psi(x^{-1} g x)$$

which sets $\psi(y) = 0$ for $y \notin H$. This is called the induction of ψ to S_n. Thus, if χ and ψ are characters of S_n and S_m respectively, we can define a multiplication in R by

$$\chi \circ \psi = (\chi \boxtimes \psi){\uparrow^{S_{n+m}}}$$

and bilinearly extending.

Let $\chi : S_n \to \Lambda_n$ and $\psi : S_n \to \Lambda_n$. Define $\langle \cdot, \cdot \rangle'$ by

$$\langle \chi, \psi \rangle' \equiv \frac{1}{n!} \sum_{g \in S_n} \chi(g) \psi(g^{-1}).$$

Since g and g^{-1} are in the same conjugacy class, if χ is a class function then

$$\mathrm{ch}_n(\chi) = \frac{1}{n!} \sum_{g \in S_n} \chi(g) p_g = \langle \chi, p \rangle'$$

where p is the function which takes $g_\lambda \in S_n$ of cycle type λ to $p_\lambda \in \Lambda_n$. It follows by Frobenius reciprocity (see for example Chapter 24 of [**BK18**]), in this setting we have

$$\langle \psi {\uparrow}^{S_n}, \chi \rangle' = \langle \psi, \chi {\downarrow}_G \rangle'.$$

This will be needed in a moment!

Definition 3.2.2: The characteristic map is:

$$\mathrm{ch} : R \longrightarrow \Lambda$$

defined to be ch_n on each degree n summand. In other words, we are defining ch to be the map $\bigoplus_{n\geq 0} \mathrm{ch}_n$ since we know that $\Lambda = \bigoplus_{n\geq 0} \Lambda^n$. This map is an isomorphism of vector spaces, and is an isometry, and in fact, its an isomorphism of algebras with respect to $\circ$.

Theorem 3.2.3 The characteristic map

$$\mathrm{ch} : R \longrightarrow \Lambda$$

is an isomorphism of algebras for the multiplication $\circ$ defined on R above.

Proof. By Theorem 3.1.4, we need only show that ch is multiplicative, that is, ch preserves the ring structure. Let χ and ψ be characters of S_n and S_m respectively. Then

$$\begin{aligned}
\mathrm{ch}(\chi \circ \psi) &= \langle \chi \circ \psi, p \rangle' \\
&= \langle (\chi \boxtimes \psi) {\uparrow}^{S_{n+m}}, p \rangle' \\
&= \langle \chi \boxtimes \psi, p {\downarrow}_{S_n \times S_m} \rangle' \\
&= \frac{1}{n!m!} \sum_{\tau\sigma \in S_n \times S_m} (\chi \boxtimes \psi)(\tau\sigma) p_{\tau\sigma} \\
&= \frac{1}{n!m!} \sum_{\tau \in S_n, \sigma \in S_m} \chi(\tau)\psi(\sigma) p_\tau p_\sigma \\
&= \left(\frac{1}{n!} \sum_{\tau \in S_n} \chi(\tau) p_\tau \right) \left(\frac{1}{m!} \sum_{\sigma \in S_m} \psi(\sigma) p_\sigma \right) \\
&= \mathrm{ch}(\chi) \cdot \mathrm{ch}(\psi).
\end{aligned}$$

□

Lemma 3.2.4: Let 1_n be the trivial character of S_n. Then $\mathrm{ch}(1_n) = h_n$.

Proof. By definition of the characteristic map and Corollary 2.4.4 we have $\mathrm{ch}(1_n) = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda = h_n$.

□

Example: If $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ is a partition of n, then $1_\lambda = 1_{\lambda_1} \circ 1_{\lambda_2} \circ \cdots$ is the character of S_n coming from the character induced by the identity character of $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots$, and since the characteristic map preserves multiplication we have

$$\mathrm{ch}(1_\lambda) = \mathrm{ch}(1_{\lambda_1} \circ 1_{\lambda_2} \circ \cdots) = \mathrm{ch}(1_{\lambda_1}) \cdot \mathrm{ch}(1_{\lambda_2}) \cdots = h_{\lambda_1} \cdot h_{\lambda_2} \cdots = h_\lambda.$$

We are now just about ready to identify the irreducible characters of the symmetric group. First, note by Corollary 2.3.9, for each partition $\lambda \vdash n$ we obtain the following determinantal formula:

$$s_\lambda = \det(h_{\lambda_i - i + j})$$

where $1 \le i \le n$ indexes rows and $1 \le j \le n$ indexes columns. Now define the character χ^λ to be

$$\chi^\lambda = \det(1_{\lambda_i - i + j})$$

where 1_λ is as above. Note that the above shows that χ^λ is a virtual character, i.e., a $\mathbb{Z}-$linear combination of irreducible characters. We now have our big theorem:

Theorem 3.2.5: The irreducible characters of S_n are exactly the collection of all χ^λ for $\lambda \vdash n$.

Proof. Since $\mathrm{ch}(1_\lambda) = h_\lambda$, and the characteristic map is an algebra isomorphism, this implies

$$\mathrm{ch}(\chi^\lambda) = \mathrm{ch}(\det(1_{\lambda_i - i + j})) = \det(h_{\lambda_i - i + j}) = s_\lambda.$$

Observe: since the Schur polynomials are orthonormal and the characteristic map is an isometry we have $\langle \chi^\lambda, \chi^\mu \rangle = \delta_{\lambda\mu}$, and since χ^λ is a virtual character with unit norm, it is therefore plus or minus an irreducible character.

Since the number of conjugacy classes in S_n is equal to the number of partitions of n, the $\{\chi^\lambda\}$ make up all of the irreducible characters (since the number of irreducible characters is the number of partitions). However, it remains to show for an irreducible character χ^λ that $-\chi^\lambda$ is not an irreducible character. It suffices to show that $\chi^\lambda(1) > 0$. By definition of the characteristic map we have

$$s_\lambda = \mathrm{ch}(\chi^\lambda) = \sum_{\mu \vdash n} z_\mu^{-1} \chi^\lambda(g_\mu) p_\mu$$

so taking the inner product $\langle \cdot, p_\mu \rangle$ of both sides, by Theorem 2.4.7 we have

$$\langle s_\lambda, p_\mu \rangle = \chi^\lambda_\mu$$

where χ^λ_μ is the value of χ^λ on an element of cycle type μ, so on the identity element we have

$$\chi^\lambda(1) = \chi^\lambda_{1^n} = \langle s_\lambda, p_1^n \rangle$$

and therefore $h_1^n = p_1^n = \sum_{\lambda \vdash n} \chi^\lambda(1) s_\lambda$. Since the $\{m_\lambda\}$ and $\{h_\lambda\}$ are dual and the Schur polynomials are self dual, the change of basis matrix from $\{h_\lambda\}$ to $\{s_\lambda\}$ has coefficients which are elements of $K_{\lambda\mu}$. In particular $\chi^\lambda(1) = K_{\lambda,1^n} > 0$ since the column 1^n is the right most column in the Kostka matrix.

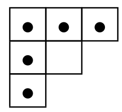
$\square$

Note that in the above proof, we stumbled upon the fact that $\chi^\lambda(1) = K_{\lambda,1^n}$, the number of semi-standard tableaux of shape λ and content $(1, 2, \ldots, n)$. Since the content is just $\{1, 2, \ldots, n\}$, this is the same as the number of standard tableaux of shape λ. This is given by the famous hook length:

$$f^\lambda = \frac{n!}{\prod h_\lambda(i,j)}$$

where f^λ is the number of standard tableau of shape λ, and $h_\lambda(i,j)$ is the "hook length" at note (i,j) in the corresponding young diagram. In particular, this formula gives the dimension of the irreducible character of S_n corresponding to partition λ under the characteristic map - so this formula allows one to compute all the dimensions of the irreducible characters rather easily.

Example: If $\lambda = (3,2,1) \vdash 6$, then the "hook" of the node (1,1) has length 5, as shown below:



In particular if we compute all the lengths and divide $n!$ by their product we obtain

$$f^\lambda = \frac{6!}{5 \cdot 3 \cdot 3 \cdot 1 \cdot 1 \cdot 1} = 16$$

which agrees with our $\lambda = (3,2,1), \mu = (1^n)$ entry of the Kostka matrix for $n = 6$ from the end of Section 2.

At this point, we are finally able to return to computing character tables of the symmetric group!

3.3 The Character Table of the Symmetric Group

Let $M(p,s)$ denote the transition matrix between the basis $\{s_\lambda\}$ and $\{p_\lambda\}$, and analogously for the other bases of Λ. That is, we denote by $M(p,s)$ the matrix $(M_{\lambda\mu})$ with rows indexed by λ and columns μ, whose coefficients come from the equation

$$p_\lambda = \sum_\mu M_{\lambda\mu} s_\mu.$$

We emphasize that in this notation $M(p,s)$, we are regarding this matrix as acting on row vectors by right multiplication.

Theorem 3.3.1: The character table of S_n is the transition matrix $M(p,s)$.

Proof. In our proof of Theorem 3.2.5 we showed that $\langle s_\lambda, p_\mu \rangle = \chi^\lambda_\mu$ where χ^λ was an irreducible character S_n evaluated on the class μ. This is equivalent to

$$p_\mu = \sum_\lambda \chi^\lambda_\mu s_\lambda$$

which exactly says that each column of the transition matrix $M(p,s)$ runs through all of the irreducible character values on the class μ. Thus $M(p,s)$ is the character table of S_n.

□

So, computing the character table of S_n boils down to computing the change of basis matrix between power sum symmetric functions and Schur functions. Recall that the Kostka matrix K from Section 2.5 gives the transition matrix from Schur functions to monomial symmetric functions, so

$$K = M(s,m).$$

Therefore, if we wish to compute $M(p,s)$, we must compute the matrix $L = M(p,m)$, since then:

$$M(p,s) = M(p,m) \cdot M(m,s) = LK^{-1}$$

so that the character table of S_n is just LK^{-1}. As before with the Kostka matrix, the matrix L has rows indexed by partitions λ and columns by compositions μ. Note that as mentioned before, the above does not make sense unless we think of these matrices as acting on row vectors on the right.

To compute this matrix L, first write $p_\lambda = \sum_\mu L_{\lambda\mu} m_\mu$. Since $m_\lambda = \sum_{\mu\sim\lambda} x^\mu$, we see that $L_{\lambda\mu}$ is the coefficient of x^μ in p_λ.

Example: When $n = 3$, we have possible partitions $(1,1,1), (2,1)$, and (3). Then:

$$p_{(1,1,1)} = p_1^3 = (x_1+x_2+x_3)^3 = x_1^3 + 3x_1^2x_2 + 3x_1^2x_3 + 3x_1x_2^2 + 3x_1x_3^2 + 6x_1x_2x_3 + x_2^3 + x_3^3 + 3x_2x_3^2 + 3x_2^2x_3$$

$$p_{(2,1)} = p_2p_1 = (x_1^2+x_2^2+x_3^2)(x_1+x_2+x_3) = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^3 + x_3^3 + x_2x_3^2 + x_2^2x_3$$

$$p_{(3)} = p_3 = x_1^3 + x_2^3 + x_3^3$$

One can see that for $\lambda = (1,1,1)$, when $\mu = (3)$ we have $L_{\lambda\mu} = 1$, when $\mu = (2,1)$ we have $L_{\lambda\mu} = 3$, and when $\mu = (1,1,1)$ we have $L_{\lambda\mu} = 6$. If we continue reading off coefficients in this manner, and use the lexicographic ordering on our rows and columns, we obtain

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 6 \end{pmatrix}$$

so this is our L matrix for $n = 3$. We can see that in this case L is lower triangular, and in fact, L is lower triangular in general (we won't prove this). If we take the inverse our Kostka matrix for $n = 3$ from the end of Section 2 and multiply on the left by L we obtain:

$$LK^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{pmatrix}$$

After taking transpose and swapping some rows and columns around, we find that:

$$LK^{-1} \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{pmatrix}$$

which matches our character table of S_3 from Section 1! The fact that we had to shuffle some rows and columns around and take transpose might bother you. It shouldn't; the transpose is simply because we prefer to write character tables with the characters on the rows, and in our $M(p,s)$ notation, these matrices were acting on row vectors on the right instead of our usual column vector notation, and as a result the initial LK^{-1} has its characters in columns instead (note the trivial character in the left most column.) The row and column operations should be more obvious: our initial character tables were not listed in lexicographic ordering so that our columns were out of order. Furthermore, from the characteristic map we obtained a particular partition λ corresponding to each character χ^λ, and back when we made our character tables our characters were not indexed with respect to this correspondence. One can check that if you make these changes to our old character table of S_3, you recover the initial LK^{-1} matrix. It is easy to tell a computer to compute the coefficients $\{L_{\lambda\mu}\}$, so let's list the L matrices for $n = 4, 5, 6$ and work out LK^{-1} using our old Kostka matrices from the end of Section 2:

$$n=4: \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 2 & 2 & 2 & 0 \\ 1 & 4 & 6 & 12 & 24 \end{pmatrix} \quad LK^{-1} = \begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 2 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 3 & 2 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 3 & 1 & 0 & -1 & -1 \\ 3 & -1 & 0 & 1 & -1 \\ 2 & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$n=5: \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 2 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 2 & 0 & 0 \\ 1 & 3 & 4 & 6 & 6 & 6 & 0 \\ 1 & 5 & 10 & 20 & 30 & 60 & 120 \end{pmatrix} \quad LK^{-1} = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 & -1 & 1 & -1 \\ 1 & 1 & -1 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & -2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 & -1 & -2 & -1 \\ 1 & 4 & 5 & 6 & 5 & 4 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 4 & 2 & 0 & 1 & -1 & -1 & 0 \\ 4 & -2 & 0 & 1 & 1 & -1 & 0 \\ 5 & 1 & 1 & -1 & 1 & 0 & -1 \\ 5 & -1 & 1 & -1 & -1 & 0 & 1 \\ 6 & 0 & -2 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$n=6: \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 6 & 2 & 3 & 6 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 1 & 2 & 3 & 2 & 4 & 4 & 0 & 6 & 4 & 0 & 0 \\ 1 & 4 & 7 & 12 & 8 & 16 & 24 & 18 & 24 & 24 & 0 \\ 1 & 6 & 15 & 30 & 20 & 60 & 120 & 90 & 180 & 360 & 720 \end{pmatrix}$$

$$LK^{-1} = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 & -1 & 0 & 0 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 & 2 & -2 & 1 & 2 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & -1 \\ 1 & 2 & 0 & 1 & -1 & -2 & 1 & -1 & 0 & 2 & 1 \\ 1 & -1 & 3 & -2 & -3 & 0 & 2 & 3 & -3 & 1 & -1 \\ 1 & 1 & 1 & -2 & 1 & 0 & -2 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 2 & 1 & 0 & -2 & -1 & -3 & -3 & -1 \\ 1 & 5 & 9 & 10 & 5 & 16 & 10 & 5 & 9 & 5 & 1 \end{pmatrix}$$

In the case of $n = 4, 5$, the matrix we obtain after properly ordering the rows and columns matches our character tables for S_4 and S_5 in Section 1! For $n = 6$, we did not make the character table in Section 1 because $n = 5$ had already became an arduous task - but here it is above now using these L and K matrices. For a sanity check, note that reading off the bottom row we have

$$1^2 + 5^2 + 9^2 + 10^2 + 5^2 + 16^2 + 10^2 + 5^2 + 9^2 + 5^2 + 1^2 = 720 = |S_6|$$

as we would hope.

As stated before, the calculation of the L and K matrices, as well as the direct calculation of the character table S_n, is made easy with the help of a computer, namely a few lines of code in Sage:

[2]:
```
Sym = SymmetricFunctions(QQ)
s = Sym.schur()
m = Sym.monomial()
p = Sym.power()

#s,m,p symmetric functions

from sage.combinat.sf.sfa import zee

#z lambda formula
sage.matrix.constructor.Matrix()

def oppdiag(num):
    table = []
    for i in range(num):
        row = [0]*(num-1)
        row.insert(num-i-1, 1)
        table.append(row)
    diag = Matrix(table)
    return(diag)

#makes opposite diagonal matrix


def CharacterTable(n):
     num = len(Partitions(n))
     diag = oppdiag(num)
     table = []
     for la in Partitions(n):
         row = []
         for mu in Partitions(n):
             row.append(s(la).scalar(p(mu)))
         table.append(row)
     char = Matrix(table)*diag
     return(char)

#makes character table of S_n



def L(n):
    table = []
    for la in Partitions(n):
        row = []
```

```
        for mu in Partitions(n):
            row.append(m(la).scalar(p(mu))*zee(mu)^(-1))
        table.append(row)
    L = Matrix(table)
    lower = L.inverse()
    return(lower)

#makes L matrix


def K(n):
     table = []
     for la in Partitions(n):
         row = []
         for mu in Partitions(n):
             row.append(m(la).scalar(s(mu)))
         table.append(row)
     K = Matrix(table)
     kostka = K.inverse()
     return(kostka)

#makes Kostka matrix







#s.transition_matrix(m,5)
#p.transition_matrix(m,5)
#the above also give the desired transitions maps


print(K(6))

print("---------------")

print(L(6))

M = L(6)*K(6)^(-1)

print("---------------")

print(M)


print("---------------")

print(CharacterTable(6))
```

```
[ 1  1  1  1  1  1  1  1  1  1  1]
[ 0  1  1  2  1  2  3  2  3  4  5]
[ 0  0  1  1  1  2  3  3  4  6  9]
```

```
[ 0  0  0  1  0  1  3  1  3  6 10]
[ 0  0  0  0  1  1  1  1  2  3  5]
[ 0  0  0  0  0  1  2  2  4  8 16]
[ 0  0  0  0  0  0  1  0  1  4 10]
[ 0  0  0  0  0  0  0  1  1  2  5]
[ 0  0  0  0  0  0  0  0  1  3  9]
[ 0  0  0  0  0  0  0  0  0  1  5]
[ 0  0  0  0  0  0  0  0  0  0  1]
--------------
[  1   0   0   0   0   0   0   0   0   0   0]
[  1   1   0   0   0   0   0   0   0   0   0]
[  1   0   1   0   0   0   0   0   0   0   0]
[  1   2   1   2   0   0   0   0   0   0   0]
[  1   0   0   0   2   0   0   0   0   0   0]
[  1   1   1   0   2   1   0   0   0   0   0]
[  1   3   3   6   2   3   6   0   0   0   0]
[  1   0   3   0   0   0   0   6   0   0   0]
[  1   2   3   2   4   4   0   6   4   0   0]
[  1   4   7  12   8  16  24  18  24  24   0]
[  1   6  15  30  20  60 120  90 180 360 720]
--------------
[ 1 -1  0  1  0  0 -1  0  0  1 -1]
[ 1  0 -1  0  0  1  0  0 -1  0  1]
[ 1 -1  1  0 -1  0  0 -1  1 -1  1]
[ 1  1 -1  0 -1  0  0  1  1 -1 -1]
[ 1 -1  0  1  2 -2  1  2  0 -1  1]
[ 1  0  0 -1  1  0  1 -1  0  0 -1]
[ 1  2  0  1 -1 -2  1 -1  0  2  1]
[ 1 -1  3 -2 -3  0  2  3 -3  1 -1]
[ 1  1  1 -2  1  0 -2  1  1  1  1]
[ 1  3  3  2  1  0 -2 -1 -3 -3 -1]
[ 1  5  9 10  5 16 10  5  9  5  1]
--------------
[ 1  1  1  1  1  1  1  1  1  1  1]
[ 5  3  1 -1  2  0 -1  1 -1  0 -1]
[ 9  3  1  3  0  0  0 -1  1 -1  0]
[10  2 -2 -2  1 -1  1  0  0  0  1]
[ 5  1  1 -3 -1  1  2 -1 -1  0  0]
[16  0  0  0 -2  0 -2  0  0  1  0]
[10 -2 -2  2  1  1  1  0  0  0 -1]
[ 5 -1  1  3 -1 -1  2  1 -1  0  0]
[ 9 -3  1 -3  0  0  0  1  1 -1  0]
[ 5 -3  1  1  2  0 -1 -1 -1  0  1]
[ 1 -1  1 -1  1 -1  1 -1  1  1 -1]
```

The $K(n)$ and $L(n)$ functions make the L and K matrices, which can also be obtained by just using the commented out transition matrix commands. The character table function creates the character table. It aligns the characters in rows and reorders the columns in reverse lexicographic order so that the degree appears on the far left. The rows are in lexicographic order based on the corresponding partitions for each character inherited from the characteristic map. For fun, here is the character table of S_7 using our Sage code:

```
[ 1  1  1  1  1  1  1  1  1  1  1  1  1  1  1]
[ 6  4  2  0  3  1 -1  0  2  0 -1  1 -1  0 -1]
[14  6  2  2  2  0  2 -1  0  0  0 -1  1 -1  0]
[15  5 -1 -3  3 -1 -1  0  1 -1  1  0  0  0  1]
[14  4  2  0 -1  1 -1  2 -2  0  1 -1 -1  0  0]
[35  5 -1  1 -1 -1 -1 -1 -1  1 -1  0  0  1  0]
[20  0 -4  0  2  0  2  2  0  0  0  0  0  0 -1]
[21  1  1 -3 -3  1  1  0 -1 -1 -1  1  1  0  0]
[21 -1  1  3 -3 -1  1  0  1 -1  1  1 -1  0  0]
[35 -5 -1 -1 -1  1 -1 -1  1  1  1  0  0 -1  0]
[15 -5 -1  3  3  1 -1  0 -1 -1 -1  0  0  0  1]
[14 -4  2  0 -1 -1 -1  2  2  0 -1 -1  1  0  0]
[14 -6  2 -2  2  0  2 -1  0  0  0 -1 -1  1  0]
[ 6 -4  2  0  3 -1 -1  0 -2  0  1  1  1  0 -1]
[ 1 -1  1 -1  1 -1  1  1 -1  1 -1  1 -1 -1  1]
```

4 Closing Remarks

This material is deep, and there are lots of other remarkable results. For example, we did not discuss the Murnaghan-Nakayama Rule, which is a method of computing the values of the irreducible character χ^λ on a given conjugacy class. It is very combinatorial in nature and uses objects called "skew hooks". Another interesting result is the Littlewood-Richardson rule, which gives a combinatorial formula for computing structure constants of Λ in the Schur functions basis. That is, it gives the coefficients of the product $s_\mu s_\nu$ in terms of the Schur basis:

$$s_\mu s_\nu = \sum_\lambda LR^\lambda_{\mu\nu} s_\lambda$$

where the $LR^\lambda_{\mu\nu}$ are the Littlewood-Richardson coefficients. Another interesting set of coefficients are the Kronecker coefficients. These are the coefficients in the decomposition of the tensor product:

$$S_\mu \otimes S_\nu = \sum_\lambda G^\lambda_{\mu\nu} S_\lambda.$$

where S_λ is the irreducible $\mathbb{C}S_n$ module corresponding to λ, usually referred to in the above notation as a Specht module. It is surprising that even today, 100+ years in the future, an efficient algorithm for computing the Kronecker coefficients is unknown. For more on these topics, see [**Sag01**].

References

[**Fro00**] Ferdinand Frobenius. *Über die Charaktere der symmetrischen Gruppe.* Sitzungsberichte der Koniglich Preussischen Akademie der Wissenschaften zu Berlin, 1900.

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[**Sag01**] Bruce E. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions.* Springer Science + Business Media New York, 2001.

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